Fluid Mechanics (A)

Course contents

- Fluid Properties
- Fluid Statics
- Fluid Kinematics
- Fluid Dynamics
- Flow through pipes
- Dimensionless analysis

Fluid Statics

- Pressure:

normal force exerted by a fluid per unit area

$$P = \frac{F}{A} \qquad\qquad \frac{N}{m^2} = Pa \text{ in SI system}$$

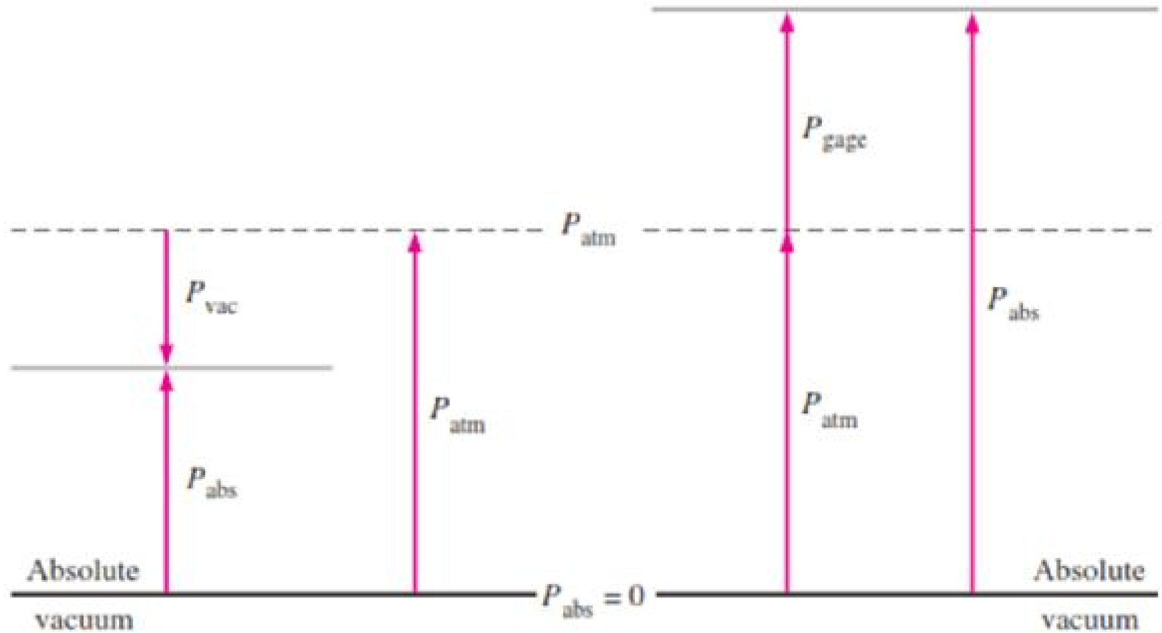
- The actual pressure at a given position is called the **absolute pressure**, and it is measured relative to absolute vacuum (i.e., absolute zero pressure).

- Most pressure-measuring devices, however, are calibrated to read zero in the atmosphere, and so they indicate the difference between the absolute pressure and the local atmospheric pressure. This difference is called the **gage pressure**.

Difference between absolute and gage pressures

$$P_{gage} = P_{abs} - P_{atm}$$

$$P_{vac} = P_{atm} - P_{abs}$$



Example: **Absolute Pressure of a Vacuum Chamber**

- A vacuum gage connected to a chamber reads 5.8 psi at a location where the atmospheric pressure is 14.5 psi. Determine the absolute pressure in the chamber.

- Solution:

$$P_{abs} = P_{atm} - P_{vac} = 14.5 - 5.8 = 8.7\ psi$$

Pressure at a point

From Newton's second law, a force balance in the *x and z directions* gives

$$\sum F_x = ma_x = 0 \qquad P_1\Delta z - P_3 l sin\theta = 0 \qquad (1)$$

$$\sum F_z = ma_z = 0 \qquad P_2\Delta x - P_3 l cos\theta - \underbrace{\frac{1}{2}\rho g \Delta x \Delta z}_{W} = 0 \qquad (2)$$

where,

ρ is the density

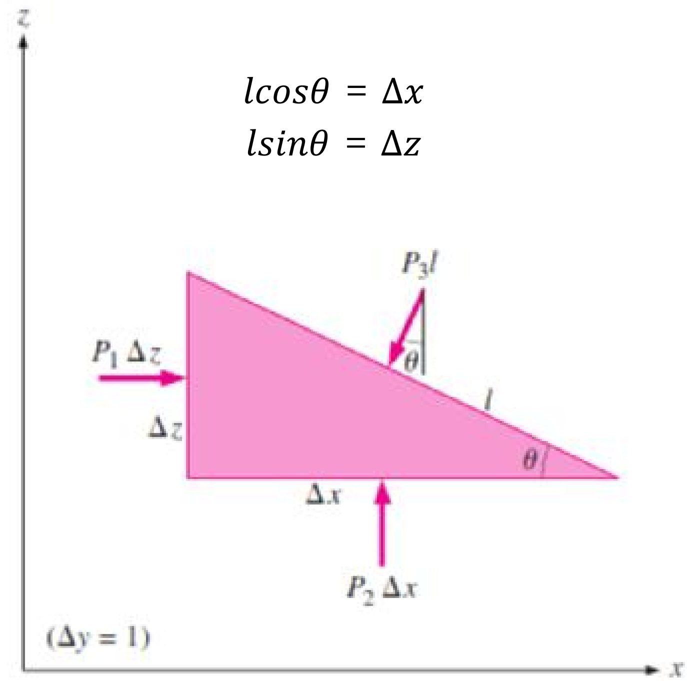
(1) $$P_1 - P_3 = 0$$

(2) $$P_2 - P_3 - \frac{1}{2}\rho g \Delta z = 0$$

By considering infinitesimal size for the wedge, Δz tends to zero

$$\therefore P_1 = P_3 = P_2 = P$$

$$lcos\theta = \Delta x$$
$$lsin\theta = \Delta z$$



Forces acting on a wedge-shaped fluid element in equilibrium.

Pressure at a point

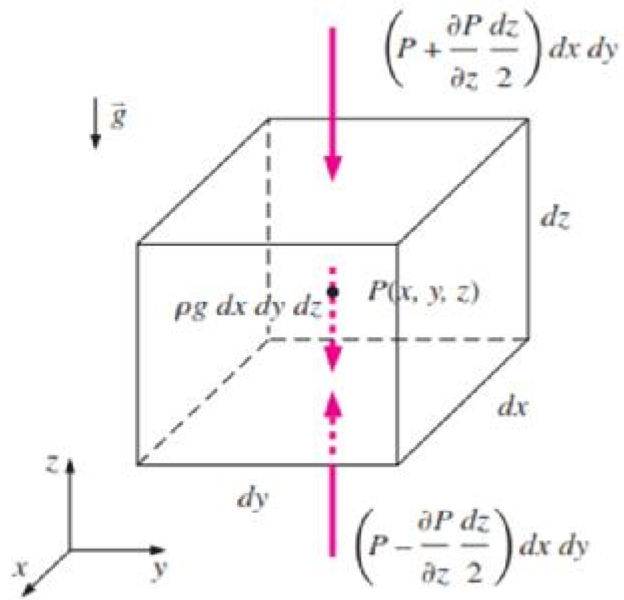
- Conclusion

The pressure at a point in a fluid has the same magnitude in all directions.

Variation of pressure with depth

- In this section we obtain relations for the pressure variation with depth in fluids in the absence of any shear stresses (i.e., no motion between fluid layers relative to each other).
- Consider a differential rectangular fluid element of side lengths *dx, dy,* and *dz.*

Newton's second law of motion for this element can be expressed as



$$\delta\vec{F} = \delta m\,.\vec{a}$$

Where,

$\delta m = \rho\, dV = \rho\, dx\, dy\, dz$	mass of the fluid element
$\delta\overrightarrow{F}$	net force acting on the element
$\overrightarrow{a}$	acceleration

Variation of pressure with depth

- Taking the pressure at the center of the element to be *P, the pressures at* the top and bottom surfaces of the element can be expressed as

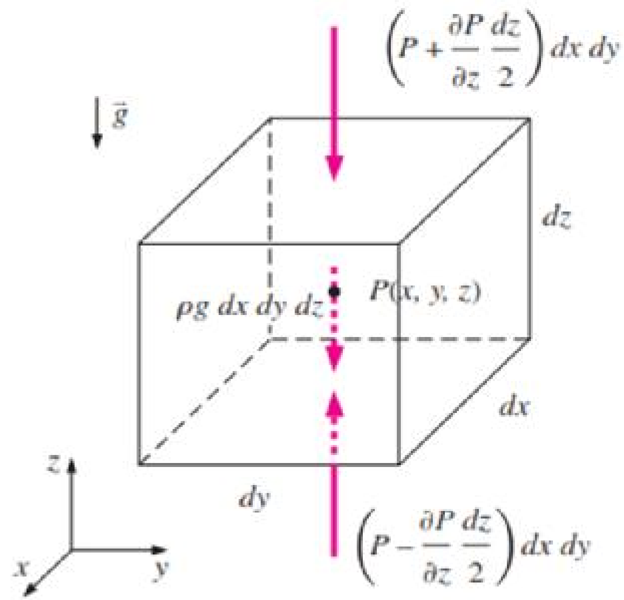
$$P + \frac{dP}{dZ}\frac{dZ}{2} \quad \text{and} \quad P - \frac{dP}{dZ}\frac{dZ}{2}$$

The net surface force acting on the element in the *z-direction:*

$$\delta F_{S,z} = \left(P - \frac{\partial P}{\partial z}\frac{dz}{2}\right) dx\, dy - \left(P + \frac{\partial P}{\partial z}\frac{dz}{2}\right) dx\, dy = -\frac{\partial P}{\partial z}\, dx\, dy\, dz$$

Similarly, the net surface forces in the *x- and y-directions are*

$$\delta F_{S,x} = -\frac{\partial P}{\partial x}\, dx\, dy\, dz \quad \text{and} \quad \delta F_{S,y} = -\frac{\partial P}{\partial y}\, dx\, dy\, dz$$



Variation of pressure with depth

- Then the surface force (which is simply the pressure force) acting on the entire element can be expressed in vector form as

$$\delta\vec{F}_S = \delta F_{S,x}\vec{i} + \delta F_{S,y}\vec{j} + \delta F_{S,z}\vec{k}$$
$$= -\left(\frac{\partial P}{\partial x}\vec{i} + \frac{\partial P}{\partial y}\vec{j} + \frac{\partial P}{\partial z}\vec{k}\right) dx\, dy\, dz = -\vec{\nabla}P\, dx\, dy\, dz$$

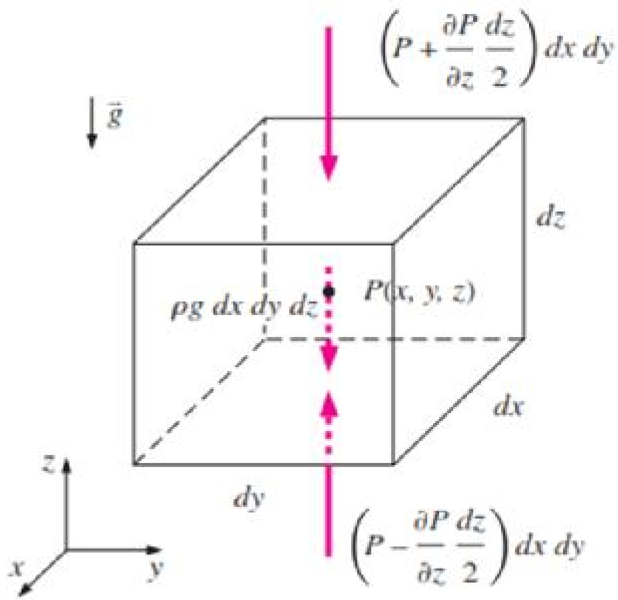
where

$$\vec{\nabla}P = \frac{\partial P}{\partial x}\vec{i} + \frac{\partial P}{\partial y}\vec{j} + \frac{\partial P}{\partial z}\vec{k}$$ is the pressure gradient

Note that: $\vec{\nabla}$ del is a vector operator that is used to express the gradients of a scalar function.

The only body force acting on the fluid element is the weight of the element

$$\delta\vec{F}_{B,z} = -g\delta m\vec{k} = -\rho g\, dx\, dy\, dz\vec{k}$$



Variation of pressure with depth

- Then the total force acting on the element becomes

$$\delta \vec{F} = \delta \vec{F}_S + \delta \vec{F}_B = -(\vec{\nabla} P + \rho g \vec{k})\, dx\, dy\, dz$$

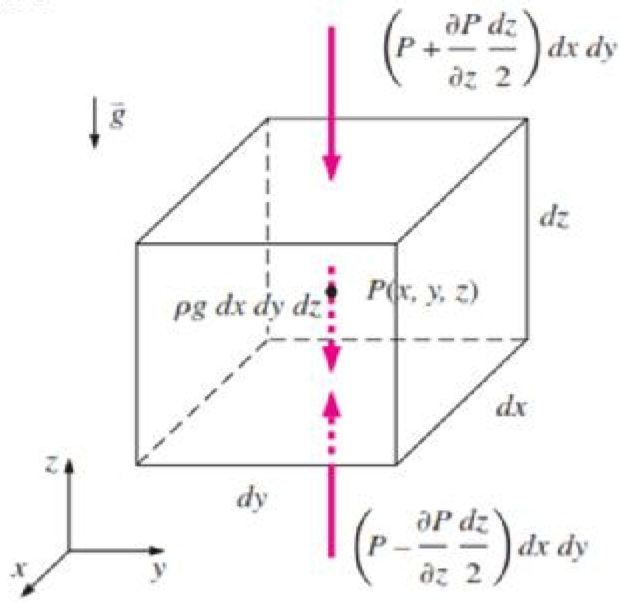
Substituting into Newton's second law of motion $\delta \vec{F} = \delta m \cdot \vec{a}$

$$\rho\, dx\, dy\, dz \,. \vec{a} = -(\vec{\nabla} P + \rho g \vec{k}) dx\, dy\, dz$$

Therefore, the general equation of motion **for a fluid that** acts as a rigid body (no shear stresses) is determined to be

$$\vec{\nabla} P + \rho g \vec{k} = -\rho \vec{a}$$

$$\frac{\partial P}{\partial x}\vec{i} + \frac{\partial P}{\partial y}\vec{j} + \frac{\partial P}{\partial z}\vec{k} + \rho g \vec{k} = -\rho(a_x \vec{i} + a_y \vec{j} + a_z \vec{k})$$



The scalar form in the three orthogonal directions

Accelerating fluids: $\frac{\partial P}{\partial x} = -\rho a_x$, $\frac{\partial P}{\partial y} = -\rho a_y$, and $\frac{\partial P}{\partial z} = -\rho(g + a_z)$

Variation of pressure with depth

- **Special Case 1: Fluids at Rest**

For fluids at rest or moving on a straight path at constant velocity, all components of acceleration are zero

$$\textit{Fluids at rest:} \qquad \frac{\partial P}{\partial x} = 0, \quad \frac{\partial P}{\partial y} = 0, \quad \text{and} \quad \frac{dP}{dz} = -\rho g$$

The pressure remains constant in any horizontal direction (*P is independent of x and y) and varies only in the vertical* direction as a result of gravity

Variation of pressure with depth

- The pressure in a fluid at rest doesn't change in horizontal direction.

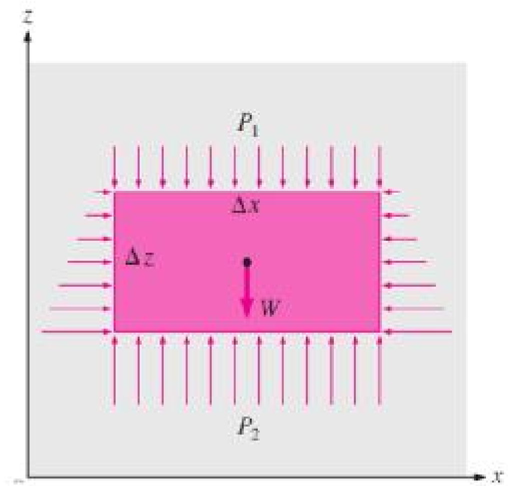
- Pressure in a fluid increases with depth because more fluid rests on deeper layers.

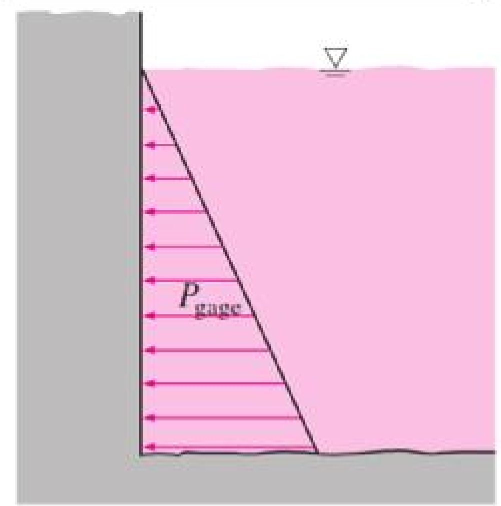
- To obtain a relation for the variation of pressure with depth, consider a rectangular fluid element of height Δz, *length* Δx, *and unit depth (into the* board) in equilibrium

- By making forces balance in vertical direction Z:

- $\sum F_Z = ma_z = 0 \rightarrow \qquad P_2\Delta x - P_1\Delta x - \rho g\Delta x\Delta z = 0$

- $\Delta P = P_2 - P_1 = \rho g\Delta z = \gamma\Delta z$
- Where, γ: specific weight of the fluid



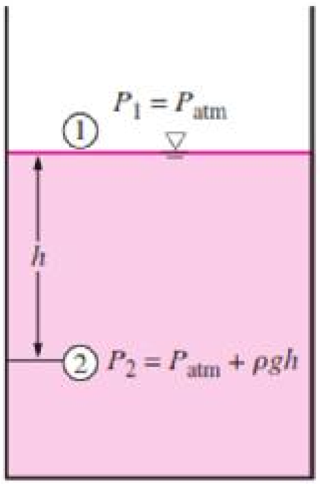


Variation of pressure with depth

- We can conclude that:

- The variation of pressure with height is negligible for gases for small to moderate distances because of their low density.
- If we take point 1 to be at the free surface of a liquid open to the atmosphere where the pressure is the atmospheric pressure P_{atm}, then the pressure at a depth *h from the free surface becomes*

$$P = P_{atm} + \rho g h \qquad\qquad P_{gage} = \rho g h$$



- For fluids whose density changes significantly with elevation

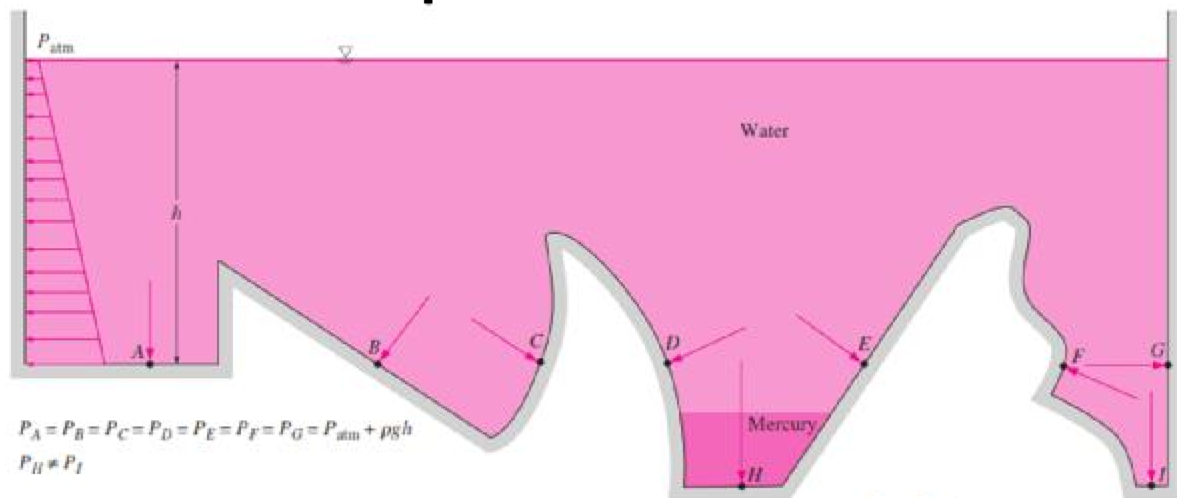
Divide eq. $P_2\Delta x - P_1\Delta x - \rho g \Delta x \Delta z = 0$ by $\Delta x \Delta z$ and taking limit as $\Delta z \to 0$

$$\therefore \frac{dP}{dz} = -\rho g$$

The negative sign is due to our taking the positive *z direction to be upward* so that *dP is negative when dz is positive since pressure decreases in an* upward direction.

Variation of pressure with depth

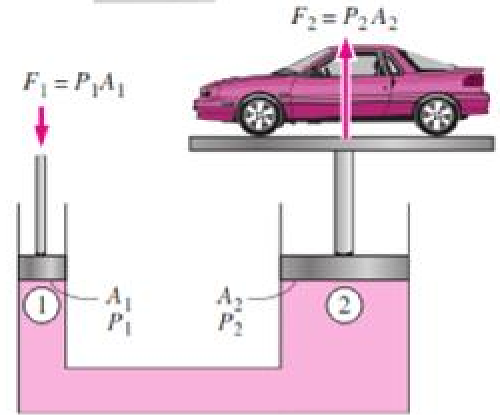
- Pressure in a fluid at rest is independent of the shape or cross section of the container.
- It changes with the vertical distance, but remains constant in other directions.
- Therefore, the pressure is the same at all points on a horizontal plane in a given fluid.



- Pascal's law:

  The pressure applied to a confined fluid increases the pressure throughout by the same amount.

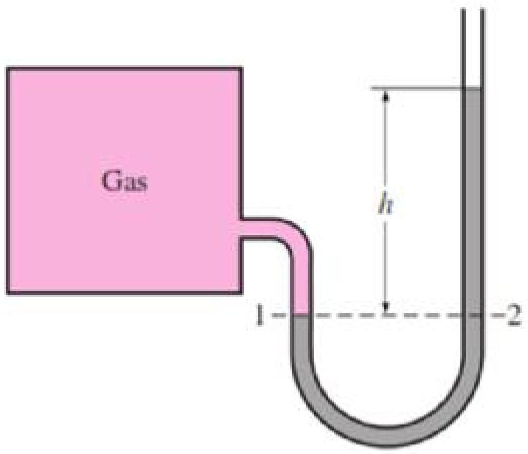
$$P_1 = P_2 \quad \rightarrow \quad \frac{F_1}{A_1} = \frac{F_2}{A_2} \quad \rightarrow \quad \frac{F_2}{F_1} = \frac{A_2}{A_1}$$



Pressure measuring devices

- Manometer:

Device uses the height of fluid column to measure pressure difference.



- The pressure at point 2 is the same as the pressure at point 1

$$P_2 = P_1$$

$$\therefore P_2 = P_{atm} + \rho g h$$ (absolute pressure)

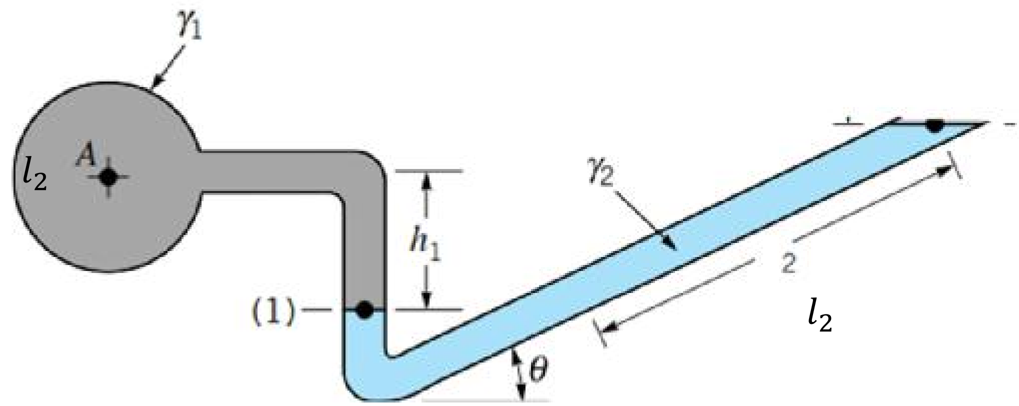
Pressure measuring devices

- Inclined tube manometer:

$$P_A + \gamma_1 h_1 - \gamma_2 l_2 sin\theta = P_{atm}$$

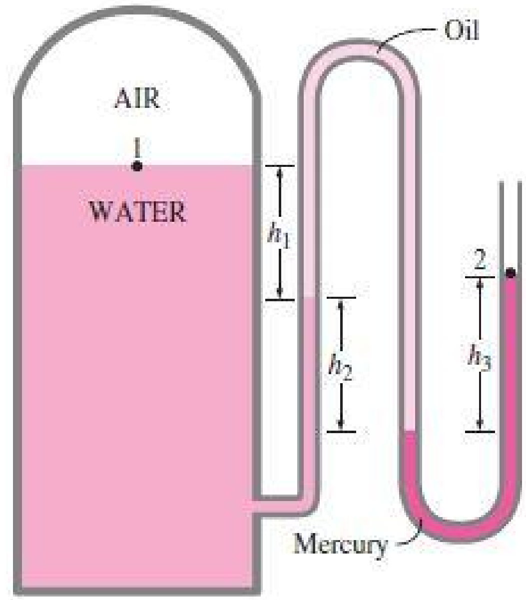
If $\gamma_1 = \gamma_2$ and $P_{atm} = 0\ (gage)$

$$\therefore\ l_2 = \frac{P_A}{\gamma_2\ sin\theta}$$



Solved Example:

- The water in a tank is pressurized by air, and the pressure is measured by a multi-fluid manometer as shown in figure. Determine the gage pressure of air in the tank if h_1= 0.2 m, h_2 = 0.3 m, and h_3 = 0.46 m. Take the densities of water, oil, and mercury to be 1000 kg/m^3, 850 kg/m3, and 13600 kg/m^3, respectively



Solution

Analysis Starting with the pressure at point 1 at the air-water interface, and moving along the tube by adding (as we go down) or subtracting (as we go up) the ρgh terms until we reach point 2, and setting the result equal to P_{atm} since the tube is open to the atmosphere gives

$$P_1 + \rho_{water} g h_1 + \rho_{oil} g h_2 - \rho_{mercury} g h_3 = P_{atm}$$

Solving for P_1,

$$P_1 = P_{atm} - \rho_{water} g h_1 - \rho_{oil} g h_2 + \rho_{mercury} g h_3$$

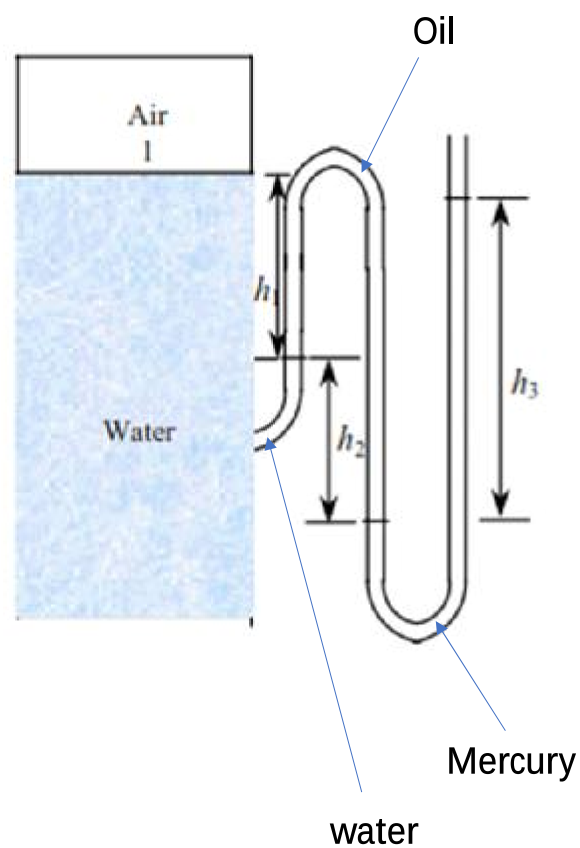
or,

$$P_1 - P_{atm} = g(\rho_{mercury} h_3 - \rho_{water} h_1 - \rho_{oil} h_2)$$

Noting that $P_{1,gage} = P_1 - P_{atm}$ and substituting,

$$P_{1,gage} = (9.81\ \text{m/s}^2)[(13{,}600\ \text{kg/m}^3)(0.46\ \text{m}) - (1000\ \text{kg/m}^3)(0.2\ \text{m}) - (850\ \text{kg/m}^3)(0.3\ \text{m})]\left(\frac{1\ \text{N}}{1\ \text{kg}\cdot\text{m/s}^2}\right)\left(\frac{1\ \text{kPa}}{1000\ \text{N/m}^2}\right)$$

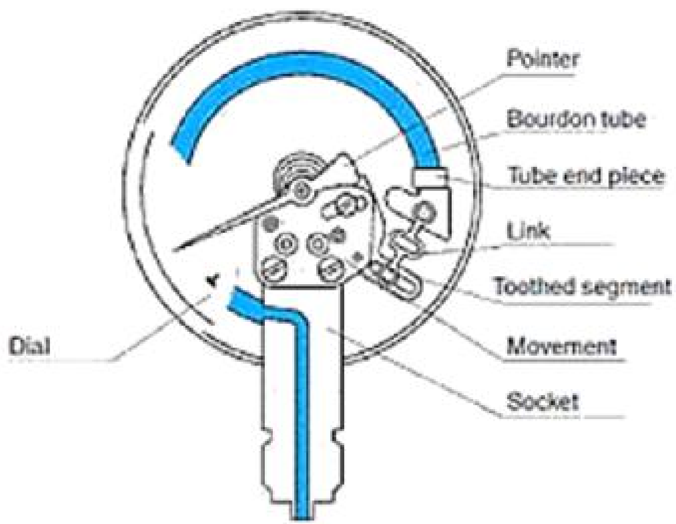
$= \textbf{56.9 kPa}$

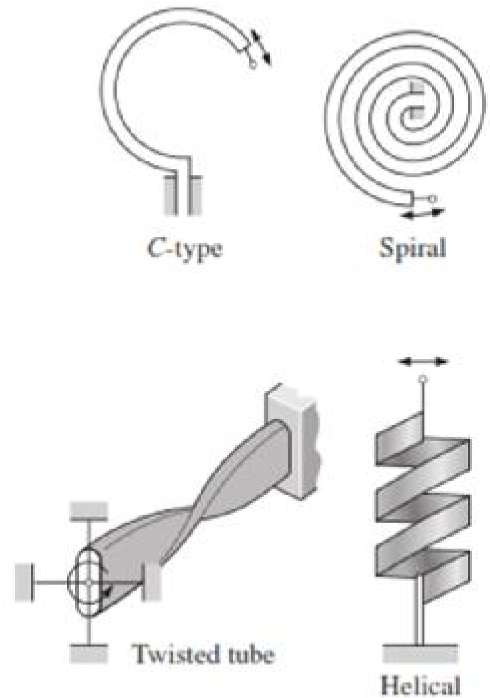


Pressure measuring devices

- Bourdon tube

    is a commonly used mechanical pressure measurement device.





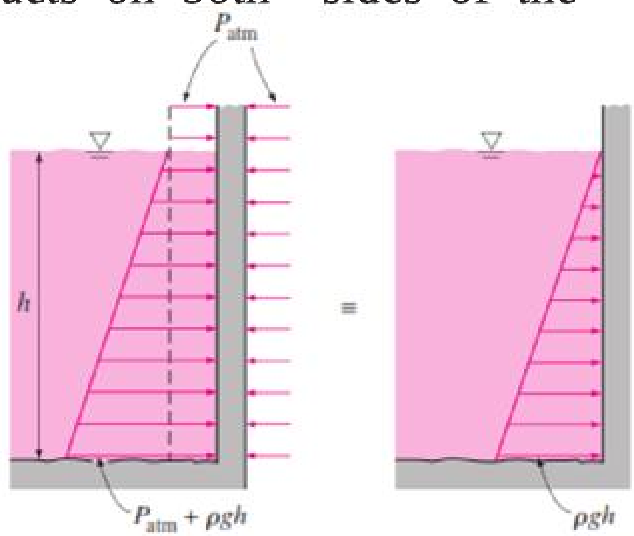
Hydrostatic forces on submerged plane surfaces

- When analyzing hydrostatic forces on submerged surfaces, the atmospheric pressure can be subtracted for simplicity when it acts on both sides of the structure.

- *We need to determine the* ***magnitude of the force and its point of application (Center of pressure)***

- In most cases, both of the gate sides are supposed to atmospheric pressure.

Therefore, it is convenient to subtract atmospheric pressure and work with gage pressure



(a) P_{atm} considered (b) P_{atm} subtracted

Hydrostatic forces on submerged plane surfaces

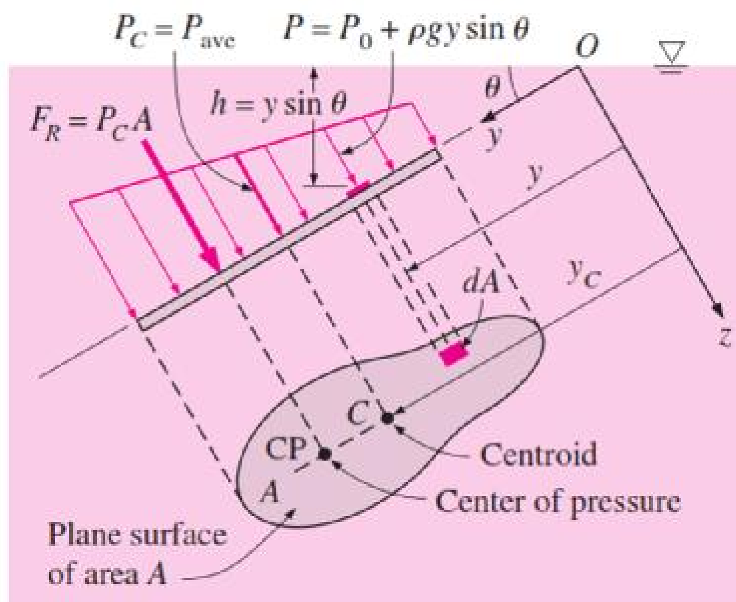
- Consider a submerged inclined plate
- The absolute pressure at any point on the plate is

$$P = P_o + \rho g h = P_o + \rho g y \ sin\theta$$

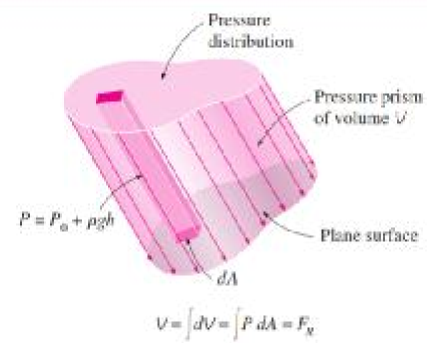
Where

h: is the vertical distance of the point from the free surface.

y: is the distance of the point from point O.



- Resultant force that acting on the plate (F_R) can be determined by integrating the force *P dA* acting on a differential area *dA* over entire surface area.



$$F_R = \int_A P\ dA = \int_A (P_o + \rho g y\ sin\theta)\ dA = P_o A + \rho g\ sin\theta \int_A y\ dA$$

Hydrostatic forces on submerged plane surfaces

$\int_A y\, dA$: is the first moment of area and related to y-coordinate of the surface centroid by

$$y_c = \frac{1}{A}\int_A y\, dA$$

By substituting,

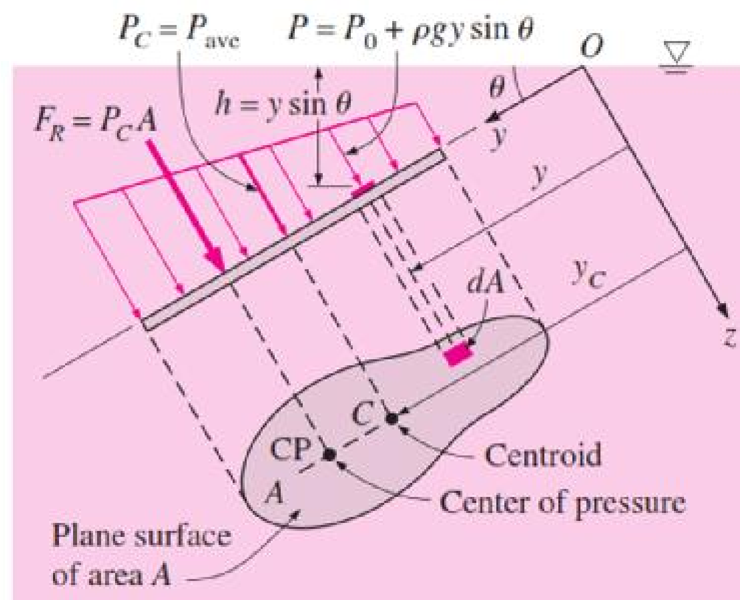
$$F_R = (P_o + \rho g y_c \sin\theta)A = (P_o + \rho g h_c)A = P_c A$$

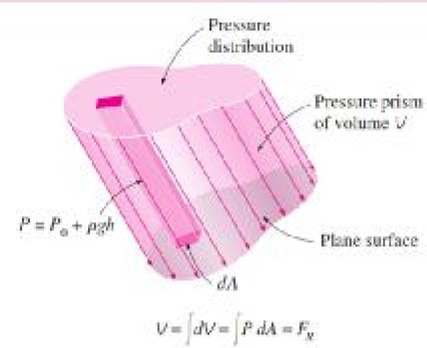
Where,

h_c : is the vertical distance of the centroid from the free surface.

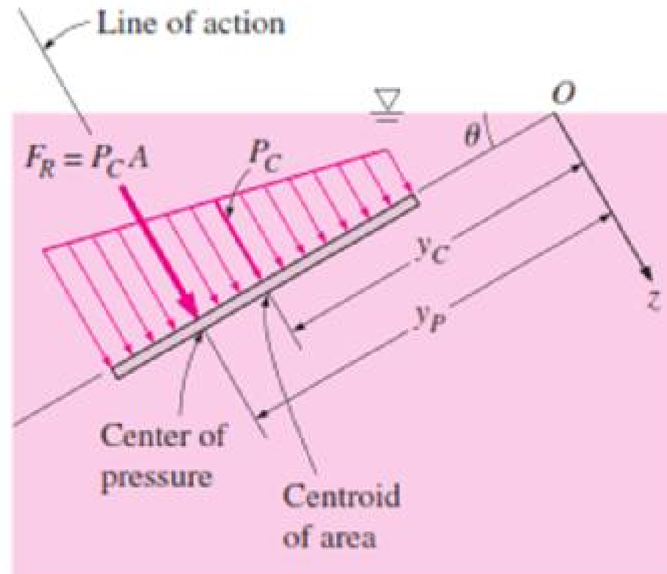
P_o: is usually atmospheric pressure, which can be ignored in most of cases.

$$F_R = \rho g h_c A$$





Hydrostatic forces on submerged plane surfaces



- Center of pressure

  We need to determine the line of action of the resultant force F_R

The vertical location of the line of action is determined by equating the moment of the resultant force to the moment of the distributed pressure force about the *x-axis. It gives*

$$y_P F_R = \int_A yP\, dA = \int_A y(\rho g y \sin\theta) dA = \rho g \sin\theta \int_A y^2\, dA$$

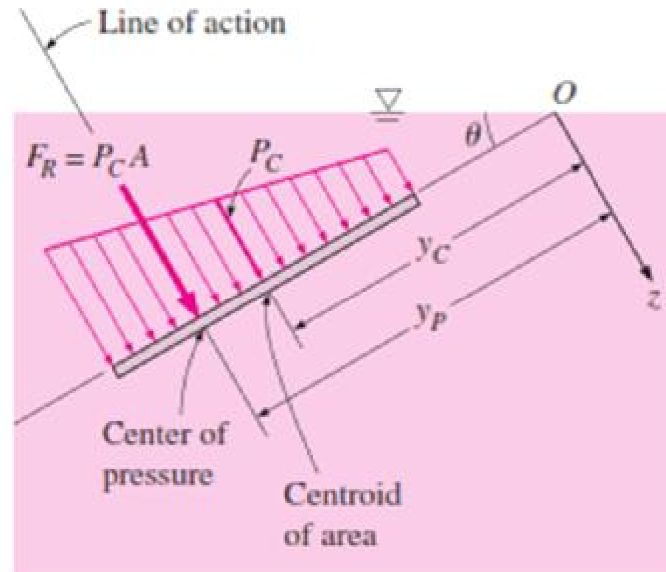
$$y_P F_R = \rho g \sin\theta\, I_{xx,o}$$

Where,

y_P: is the distance of center of pressure from x-axis (point O)

$I_{xx,o} = \int_A y^2\, dA$: is the second moment of area about x-axis passing through point O m^4

Hydrostatic forces on submerged plane surfaces



- We must get the second moment of area about the center of pressure (C).

By parallel axis theorem:

$$I_{xx,o} = I_{xx,c} + y_c^2 A$$

$I_{xx,c}$: is the second moment of area about x-axis passing through the centroid of the area m^4

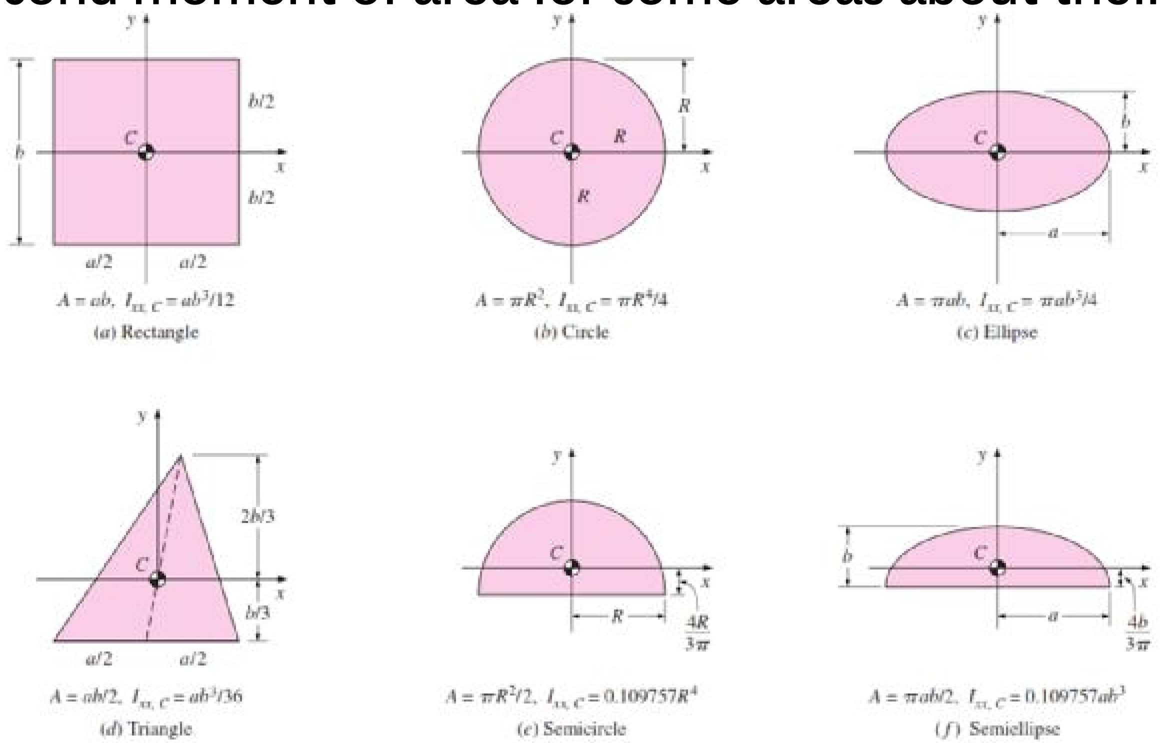
By substituting:

$$\frac{y_P F_R}{\rho g \, sin\theta} = I_{xx,c} + y_c^2 A$$

$$\frac{y_P(\rho g y_c \, sin\theta A)}{\rho g \, sin\theta} = I_{xx,c} + y_c^2 A \quad \rightarrow \quad y_P = y_c + \frac{I_{xx,c}}{y_c A}$$

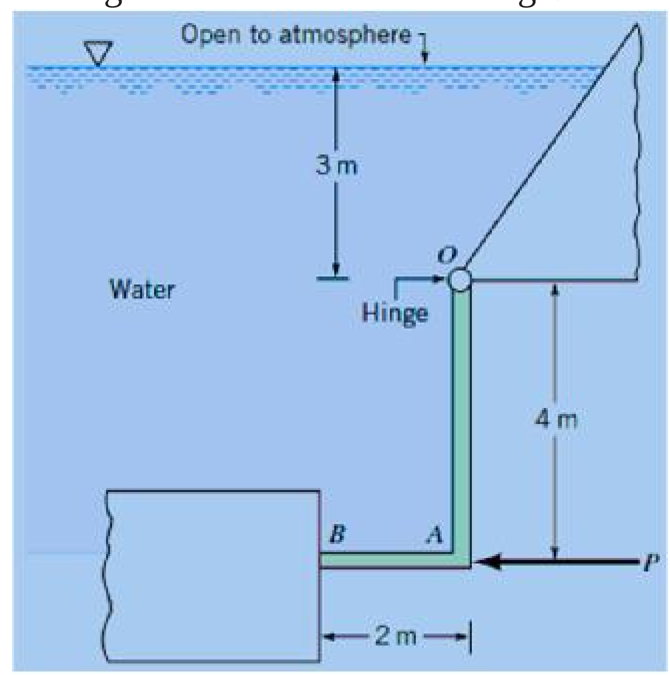
Hydrostatic forces on submerged plane surfaces

- The second moment of area for some areas about their centroids



Example on Hydrostatic forces on submerged plane surfaces

- Example :
- The rigid gate, *OAB is hinged at O and* rests against a rigid support at *B. What minimum horizontal* force, *P, is required to hold the gate closed if its width is 3 m?* Neglect the weight of the gate and friction in the hinge. The back of the gate is exposed to the atmosphere.



$F_1 = \gamma h_{c_1} A_1$ where $h_{c_1} = 5\,m$

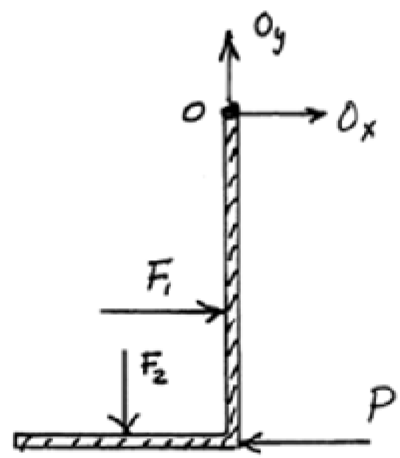
Thus, $F_1 = (9800\ \frac{N}{m^3})(5\,m)(4\,m \times 3\,m)$

$= 5.88 \times 10^5\ N$

$F_2 = \gamma h_{c_2} A_2$ where $h_{c_2} = 7\,m$

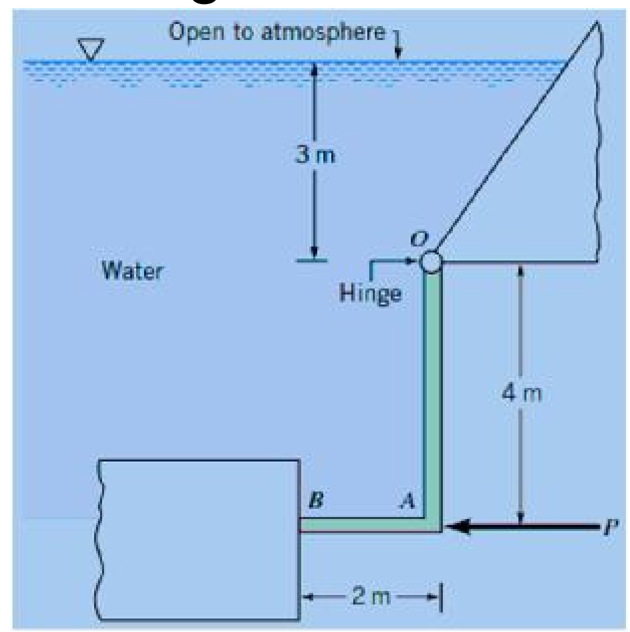
So that $F_2 = (9800\ \frac{N}{m^3})(7\,m)(2\,m \times 3\,m)$

$= 4.12 \times 10^5\ N$



Example on Hydrostatic forces on submerged plane surfaces

- Example :



To locate F_1,

$$y_{R_1} = \frac{I_{xc}}{y_{c_1} A_1} + y_{c_1} = \frac{\frac{1}{12}(3m)(4m)^3}{(5m)(4m \times 3m)} + 5m = 5.267m$$

The force F_2 acts at the center of the AB section. Thus,

$$\Sigma M_O = 0$$

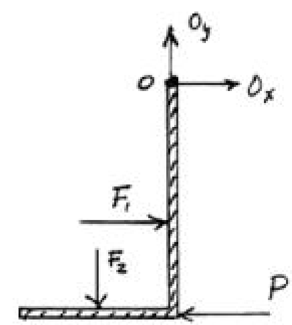
and

$$F_1 (5.267m - 3m) + F_2 (1m) = P (4m)$$

so that

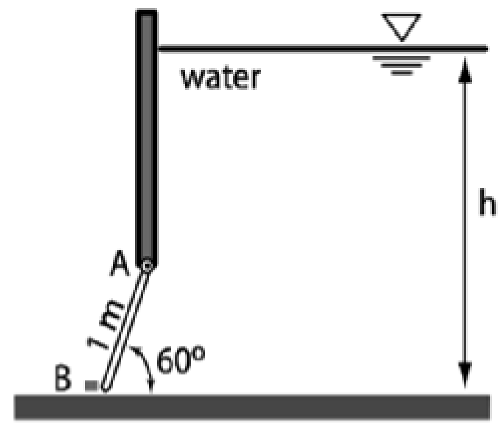
$$P = \frac{(5.88 \times 10^5 N)(2.267m) + (4.12 \times 10^5 N)(1m)}{4m}$$

$$= \underline{\underline{436 \ kN}}$$



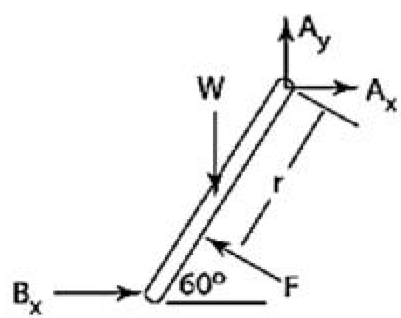
Solved example

A rectangular gate of dimension 1 m by 4 m is held in place by a stop block at B. This block exerts a horizontal force of 40 kN and a vertical force of 0 kN. The gate is pin-connected at A, and the weight of the gate is 2 kN. Find the depth h of the water.



Solution

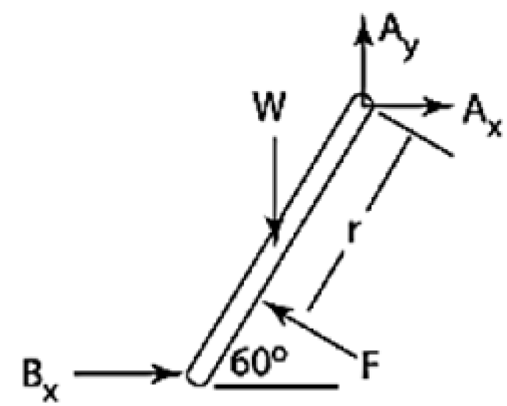
A free-body diagram of the gate is



where W is the weight of the gate, F is the equivalent force of the water, and r is the length of the moment arm. Summing moments about A gives

$$B_x(1.0 \sin 60^o) - F \times r + W(0.5 \cos 60^o) = 0$$

$$
\begin{aligned}
F \times r &= B_x \sin 60^o + W(0.5 \cos 60^o) \\
&= 40,000 \sin 60^o + 2000(0.5 \cos 60^o) \qquad (1) \\
&= 35,140 \text{ N-m}
\end{aligned}
$$



The hydrostatic force F acts at a distance $\overline{I}/\overline{y}A$ below the centroid of the plate. Thus the length of the moment arm is

$$r = 0.5 \text{ m} + \frac{\overline{I}}{\overline{y}A} \qquad (2)$$

$$y_P = y_c + \frac{I_{xx,c}}{y_c A}$$

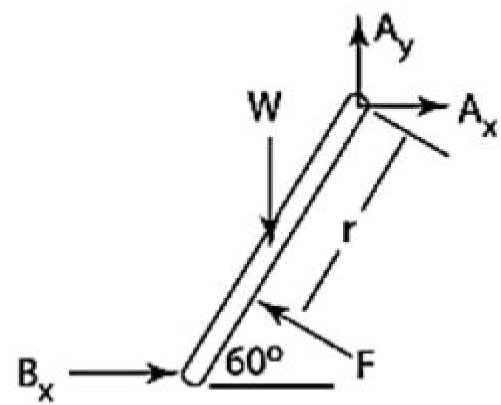
Analysis of terms in Eq. (2) gives

$$
\begin{aligned}
\overline{y} &= (h/\sin(60^o) - 0.5) \\
\overline{I} &= 4 \times 1^3/12 = 0.333 \\
A &= 4 \times 1 = 4
\end{aligned}
$$

Eq. (2) becomes

$$r = 0.5 + \frac{0.0833}{(h/\sin(60^o) - 0.5)} \qquad (3)$$

The equivalent force of the water is



$$
\begin{aligned}
F &= \overline{p}A \\
&= \gamma(h - 0.5 \sin 60^o)4 \\
&= 9,810(h - 0.5 \sin 60^o)4 \\
&= 39,240(h - 0.433)
\end{aligned}
\tag{4}
$$

Substituting Eqs. (3) and (4) into Eq. (1) gives

$$35,140 = Fr$$

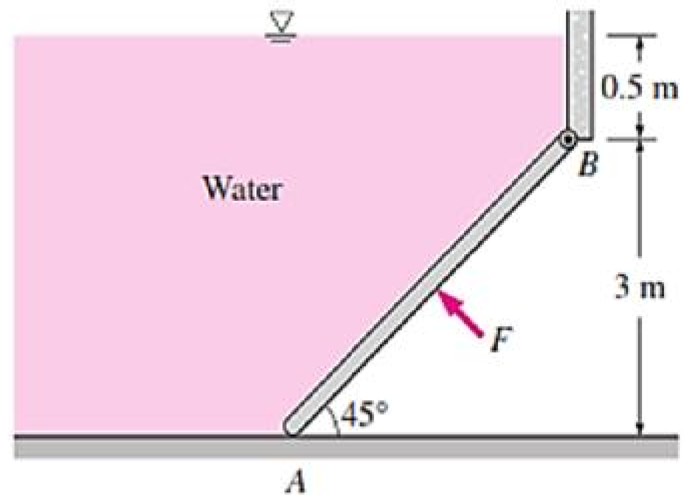
$$35,140 = [39,240(h - 0.433)] \left[0.5 + \frac{0.0833}{(1.155h - 0.5)} \right] \tag{5}$$

Eq. (5) has a single unknown (the depth of water h). To solve Eq. (5), one may use a computer program that finds the root of an equation. This was done, and the answer is

$$h = \underline{\underline{2.08 \text{ m}}}$$

Try to solve the problem

The 200-kg, 5-m-wide rectangular gate shown in the figure is hinged at *B* and leans against the floor at *A* making an angle of 45° with the horizontal. The gate is to be opened from its lower edge by applying a normal force at its center. Determine the minimum force *F* required to open the water gate.



Hydrostatic forces on submerged curved surfaces

- In order to determine F_R on two-dimensional curved surface, you have to determine the horizontal and vertical components F_H and F_V separately.
- Based on Newton's third law, the resultant force acting on a curved solid surface in equal an opposite to the force acting on the curved liquid surface.

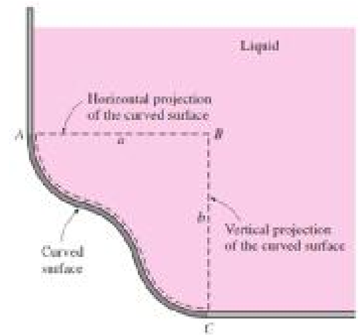
Balance of horizontal forces $\quad F_H = F_x$

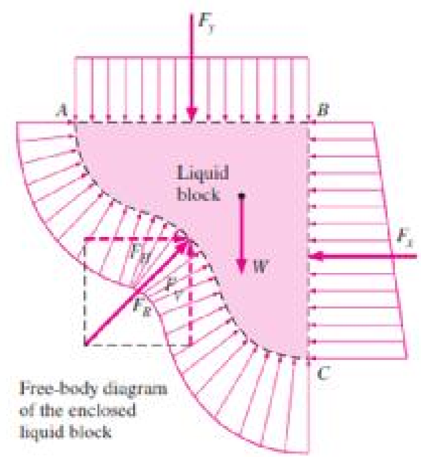
Balance of vertical forces $\quad F_V = F_y + W$

Resultant force $\quad F_R = \sqrt{F_H^2 + F_V^2}$

Angle between line of action of F_R and horizontal can be determined by

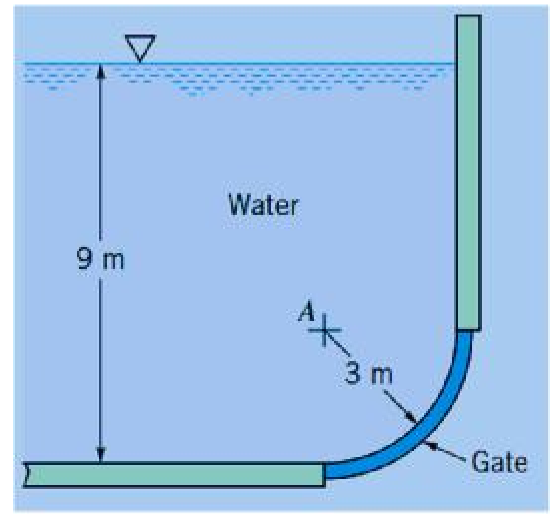
$$\tan\alpha = \frac{F_V}{F_H}$$





Example on Hydrostatic forces on submerged curved surfaces

- Example :
- A 4-m-long curved gate is located in the side of a reservoir containing water as shown in the Fig. Determine the magnitude of the horizontal and vertical components of the force of the water on the gate. Will this force pass through point *A? Explain.*



For equilibrium,

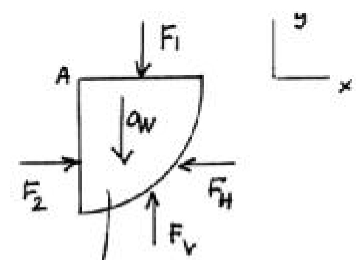
$$\Sigma F_x = 0$$

or

$$F_H = F_2 = \gamma h_{c2} A_2 = \gamma (6m + 1.5m)(3m \times 4m)$$

so that

$$F_H = \left(9.80 \frac{kN}{m^3}\right)(7.5m)(12m^2) = \underline{\underline{882\ kN}}$$



Similarly,

$$\Sigma F_y = 0$$

$$F_V = F_1 + W \qquad \text{where :}$$

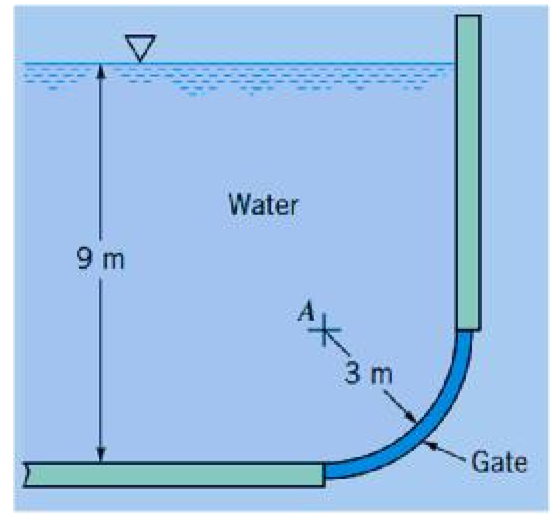
$$F_1 = [\gamma (6m)](3m \times 4m) = \left(9.80 \frac{kN}{m^3}\right)(6m)(12m^2)$$

volume ~ $\forall$

and

$$\forall = \frac{\pi (3m)^2}{4} \times 4m = 9\pi\ m^3$$

Example on Hydrostatic forces on submerged curved surfaces



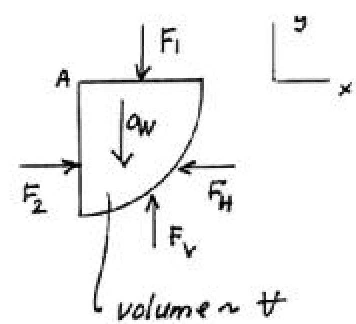
- Example :

$$\mathcal{W}_W = \gamma \forall = \left(9.80 \frac{kN}{m^3}\right)\left(9\pi\ m^3\right)$$

Thus,

$$F_V = \left(9.80 \frac{kN}{m^3}\right)\left[72\ m^3 + 9\pi\ m^3\right] = \underline{\underline{983\ kN}}$$

(Note: Force of water on gate will be opposite in direction to that shown on figure.)



and

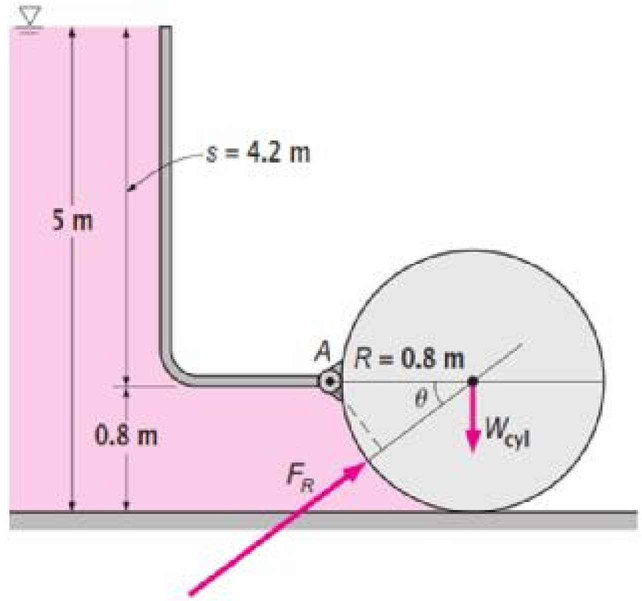
$$\forall = \frac{\pi (3\ m)^2}{4} \times 4\ m = 9\pi\ m^3$$

The direction of all differential forces acting on the curved surface is perpendicular to surface, and therefore, the resultant must pass through the intersection of all these forces which is at point A. <u>Yes.</u>

Example:

A Gravity-Controlled Cylindrical Gate

A long solid cylinder of radius 0.8 m hinged at point A is used as an automatic gate, as shown in the fig. When the water level reaches 5 m, the gate opens by turning about the hinge at point A. Determine (a) the hydrostatic force acting on the cylinder and its line of action when the gate opens and (b) the weight of the cylinder per m length of the cylinder.



$$F_H = F_x = P_{ave}A = \rho g h_C A = \rho g(s + R/2)A$$

$$= (1000\ \text{kg/m}^3)(9.81\ \text{m/s}^2)(4.2 + 0.8/2\ \text{m})(0.8\ \text{m} \times 1\ \text{m})\left(\frac{1\ \text{kN}}{1000\ \text{kg}\cdot\text{m/s}^2}\right)$$

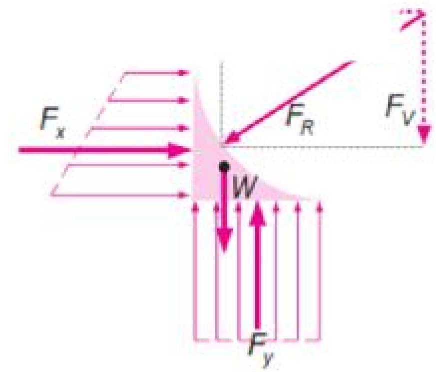
$$= 36.1\ \text{kN}$$

Vertical force on horizontal surface (upward):

$$F_y = P_{ave}A = \rho g h_C A = \rho g h_{bottom} A$$

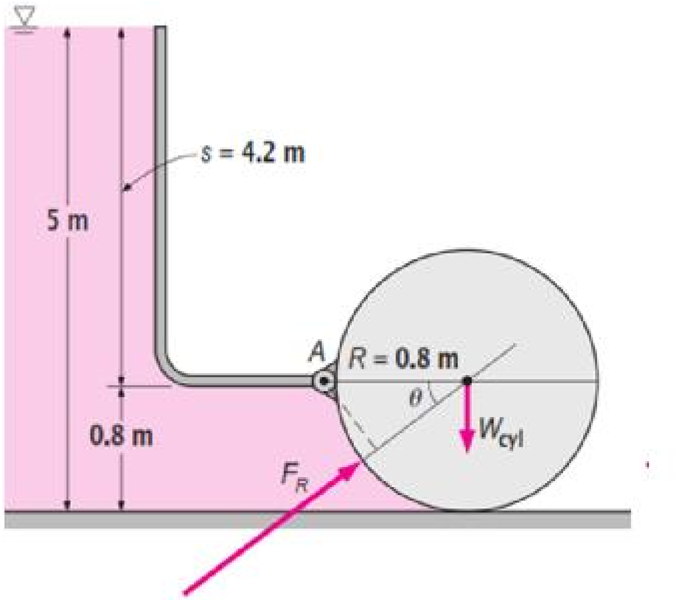
$$= (1000\ \text{kg/m}^3)(9.81\ \text{m/s}^2)(5\ \text{m})(0.8\ \text{m} \times 1\ \text{m})\left(\frac{1\ \text{kN}}{1000\ \text{kg}\cdot\text{m/s}^2}\right)$$

$$= 39.2\ \text{kN}$$



Example:

A Gravity-Controlled Cylindrical Gate



Weight of fluid block per m length (downward):

$$W = mg = \rho g V = \rho g(R^2 - \pi R^2/4)(1\ \text{m})$$

$$= (1000\ \text{kg/m}^3)(9.81\ \text{m/s}^2)(0.8\ \text{m})^2(1 - \pi/4)(1\ \text{m})\left(\frac{1\ \text{kN}}{1000\ \text{kg}\cdot\text{m/s}^2}\right)$$

$$= 1.3\ \text{kN}$$

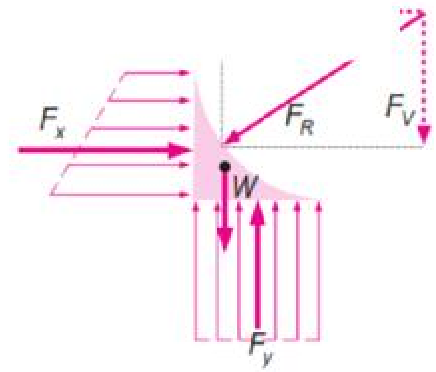
Therefore, the net upward vertical force is

$$F_V = F_y - W = 39.2 - 1.3 = 37.9\ \text{kN}$$

Then the magnitude and direction of the hydrostatic force acting on the cylindrical surface become

$$F_R = \sqrt{F_H^2 + F_V^2} = \sqrt{36.1^2 + 37.9^2} = 52.3\ \text{kN}$$

$$\tan\theta = F_V/F_H = 37.9/36.1 = 1.05 \rightarrow \theta = 46.4°$$

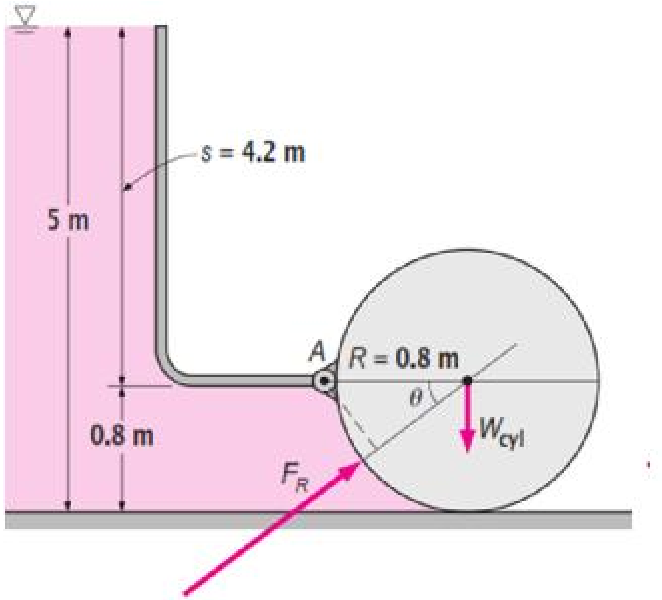


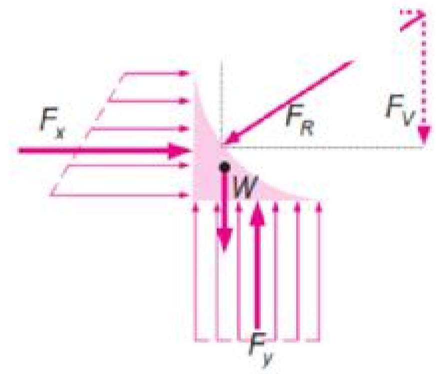
Example:

A Gravity-Controlled Cylindrical Gate

(*b*) When the water level is 5 m high, the gate is about to open and thus the reaction force at the bottom of the cylinder is zero. Then the forces other than those at the hinge acting on the cylinder are its weight, acting through the center, and the hydrostatic force exerted by water. Taking a moment about point *A* at the location of the hinge and equating it to zero gives

$$F_R R \sin\theta - W_{cyl} R = 0 \rightarrow W_{cyl} = F_R \sin\theta = (52.3\ \text{kN}) \sin 46.4° = 37.9\ \text{kN}$$





Fluids in rigid body motion

- In this section we obtain relations for the pressure variation in fluids moving like a solid body with or without acceleration in the absence of any shear stresses (i.e., no motion between fluid layers relative to each other).
- Consider a differential rectangular fluid element of side lengths *dx, dy,* and *dz.*

Newton's second law of motion for this element can be expressed as

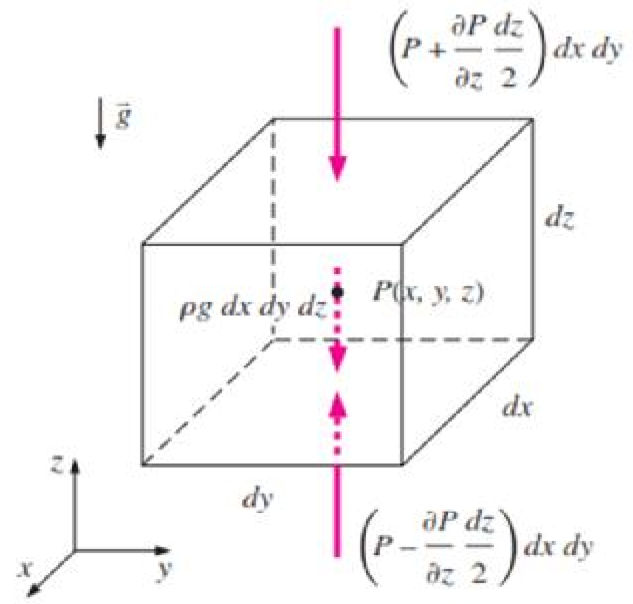
$$\delta\vec{F} = \delta m \,.\, \vec{a}$$

Where,

$\delta m = \rho\, dV = \rho\, dx\, dy\, dz$ — mass of the fluid element

$\delta\vec{F}$ — net force acting on the element

$\vec{a}$ — acceleration



Fluids in rigid body motion

- Taking the pressure at the center of the element to be *P, the pressures at* the top and bottom surfaces of the element can be expressed as

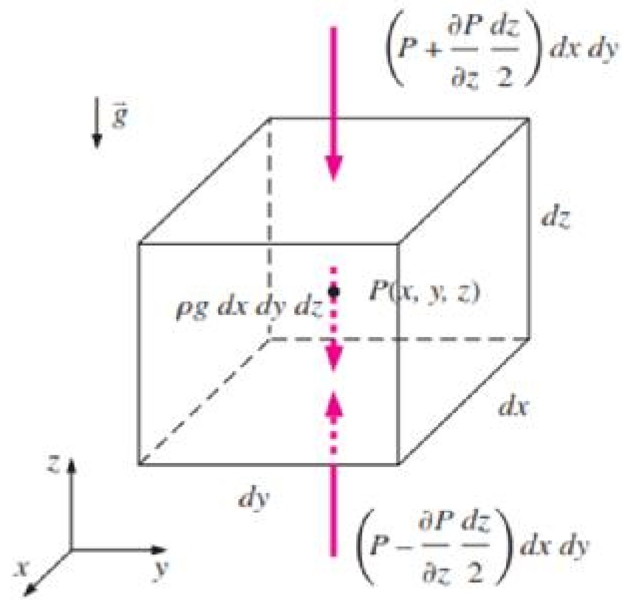
$$P + \frac{dP}{dZ}\frac{dZ}{2} \quad \text{and} \qquad P - \frac{dP}{dZ}\frac{dZ}{2}$$

The net surface force acting on the element in the *z-direction:*

$$\delta F_{S,z} = \left(P - \frac{\partial P}{\partial z}\frac{dz}{2}\right) dx\, dy - \left(P + \frac{\partial P}{\partial z}\frac{dz}{2}\right) dx\, dy = -\frac{\partial P}{\partial z} dx\, dy\, dz$$

Similarly, the net surface forces in the *x- and y-directions are*

$$\delta F_{S,x} = -\frac{\partial P}{\partial x} dx\, dy\, dz \qquad \text{and} \qquad \delta F_{S,y} = -\frac{\partial P}{\partial y} dx\, dy\, dz$$



Fluids in rigid body motion

- Then the surface force (which is simply the pressure force) acting on the entire element can be expressed in vector form as

$$\delta \vec{F}_S = \delta F_{S,x}\vec{i} + \delta F_{S,y}\vec{j} + \delta F_{S,z}\vec{k}$$
$$= -\left(\frac{\partial P}{\partial x}\vec{i} + \frac{\partial P}{\partial y}\vec{j} + \frac{\partial P}{\partial z}\vec{k}\right) dx\, dy\, dz = -\vec{\nabla} P\, dx\, dy\, dz$$

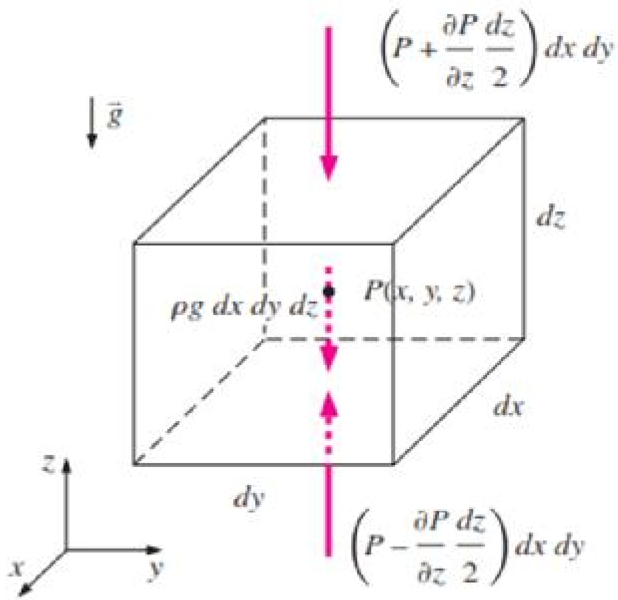
where

$$\vec{\nabla} P = \frac{\partial P}{\partial x}\vec{i} + \frac{\partial P}{\partial y}\vec{j} + \frac{\partial P}{\partial z}\vec{k}$$ is the pressure gradient

Note that: $\vec{\nabla}$ del is a vector operator that is used to express the gradients of a scalar function.

The only body force acting on the fluid element is the weight of the element

$$\delta \vec{F}_{B,z} = -g\delta m\vec{k} = -\rho g\, dx\, dy\, dz\vec{k}$$



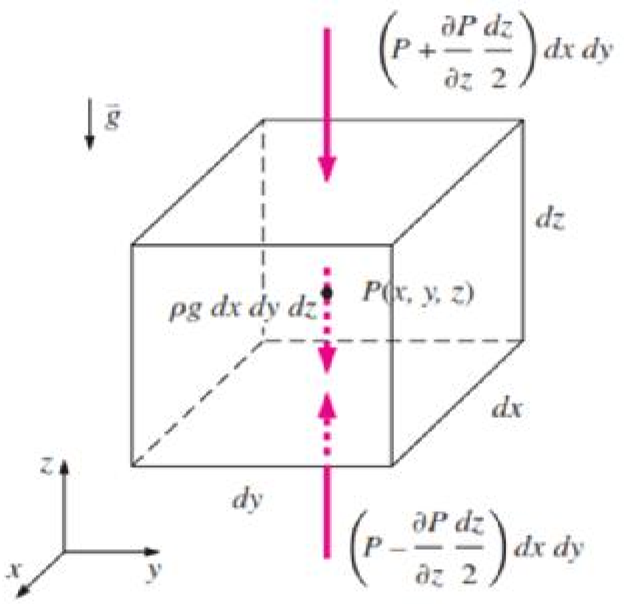
Fluids in rigid body motion

- Then the total force acting on the element becomes

$$\delta\vec{F} = \delta\vec{F}_S + \delta\vec{F}_B = -(\vec{\nabla}P + \rho g\vec{k})\, dx\, dy\, dz$$

Substituting into Newton's second law of motion $\delta\vec{F} = \delta m \cdot \vec{a}$

$$\rho\, dx\, dy\, dz\, .\vec{a} = -(\vec{\nabla}P + \rho g\vec{k}) dx\, dy\, dz$$



Therefore, the general equation of motion **for a fluid that** acts as a rigid body (no shear stresses) is determined to be

$$\vec{\nabla}P + \rho g\vec{k} = -\rho\vec{a}$$

$$\frac{\partial P}{\partial x}\vec{i} + \frac{\partial P}{\partial y}\vec{j} + \frac{\partial P}{\partial z}\vec{k} + \rho g\vec{k} = -\rho(a_x\vec{i} + a_y\vec{j} + a_z\vec{k})$$

The scalar form in the three orthogonal directions

Accelerating fluids: $\frac{\partial P}{\partial x} = -\rho a_x$, $\frac{\partial P}{\partial y} = -\rho a_y$, and $\frac{\partial P}{\partial z} = -\rho(g + a_z)$

Fluids in Rigid-body Motion

Special Case 2: Free Fall of a Fluid Body

- A freely falling body accelerates under the influence of gravity. When the air resistance is negligible, the acceleration of the body equals the gravitational acceleration, and acceleration in any horizontal direction is zero.
- Therefore, $a_x = a_y = 0$ *and* $a_z = -g$. Then *the equations of motion for* accelerating fluids reduce to

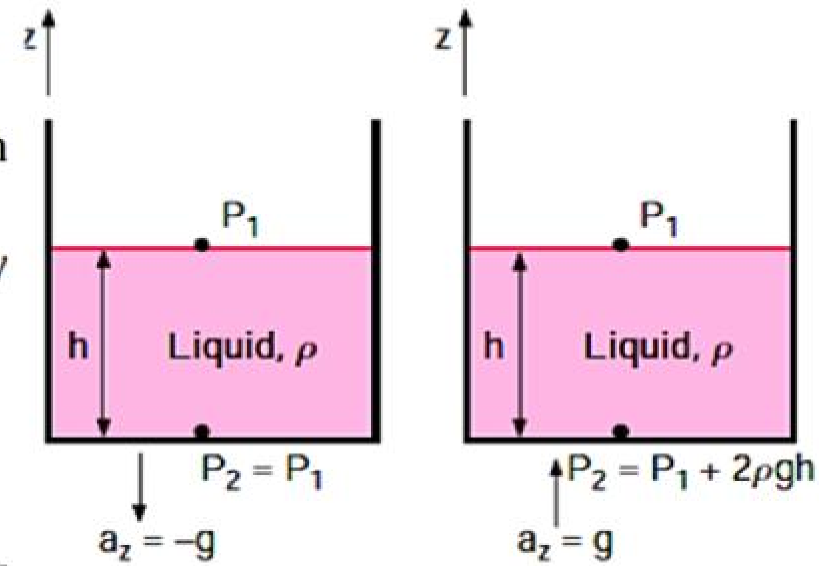
Free-falling fluids: $$\frac{\partial P}{\partial x} = \frac{\partial P}{\partial y} = \frac{\partial P}{\partial z} = 0 \quad \rightarrow \quad P = \text{constant}$$

- Therefore, in a frame of reference moving with the fluid, it behaves like it is in an environment with zero gravity. Also, the gage pressure in a drop of liquid in free fall is zero throughout.

Fluids in Rigid-body Motion

Special Case 2: Free Fall of a Fluid Body

- When the direction of motion is reversed and the fluid is forced to accelerate vertically with $a_z = +g$ by placing the fluid container in an elevator or a space vehicle propelled upward by a rocket engine, the pressure gradient in the z-direction is , $\partial P/\partial z = -2\rho g$.
- Therefore, the pressure difference across a fluid layer now doubles relative to the stationary fluid case



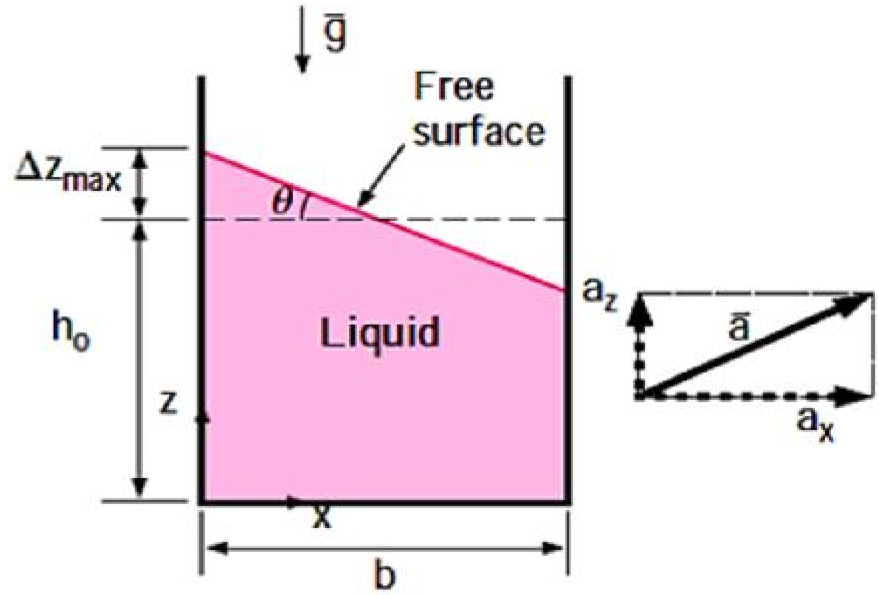
(a) Free fall of a liquid

(b) Upward acceleration of a liquid with $a_z = +g$

Fig. The effect of acceleration on the pressure of a liquid during free fall and upward acceleration.

Acceleration on a Straight Path

- Consider a container partially filled with a liquid. The container is moving on a straight path with a constant acceleration.
- We take the projection of the path of motion on the horizontal plane to be the *x-axis,* and the projection on the vertical plane to be the *z-axis*



- The *x- and z- components* of acceleration are a_x *and* a_z. There is no movement in the y-direction, and thus the acceleration in that direction is zero, $a_y = 0$.

Acceleration on a Straight Path

- Then the equations of motion for accelerating fluids reduce to

$$\frac{\partial P}{\partial x} = -\rho a_x, \quad \frac{\partial P}{\partial y} = 0, \quad \text{and} \quad \frac{\partial P}{\partial z} = -\rho(g + a_z)$$

- Therefore, pressure is independent of *y.*
- *Then the total differential of P*

  = P(x, z), which is $(\partial P/\partial x)\,dx + (\partial P/\partial z)\,dz$, becomes

$$dP = -\rho a_x\,dx - \rho(g + a_z)\,dz$$

- For ρ = constant, the pressure difference between two points 1 and 2 in the fluid is determined by integration to be

$$P_2 - P_1 = -\rho a_x(x_2 - x_1) - \rho(g + a_z)(z_2 - z_1)$$

- Taking point 1 to be the origin (*x = 0, z = 0)* where the pressure is P_0 *and* point 2 to be any point in the fluid (no subscript), the pressure distribution can be expressed as

Acceleration on a Straight Path

Pressure variation: $$P = P_0 - \rho a_x x - \rho(g + a_z)z$$

- The vertical rise (or drop) of the free surface at point 2 relative to point 1 can be determined by choosing both 1 and 2 on the free surface (so that $P_1 = P_2$), and solving for $z_2 - z_1$,

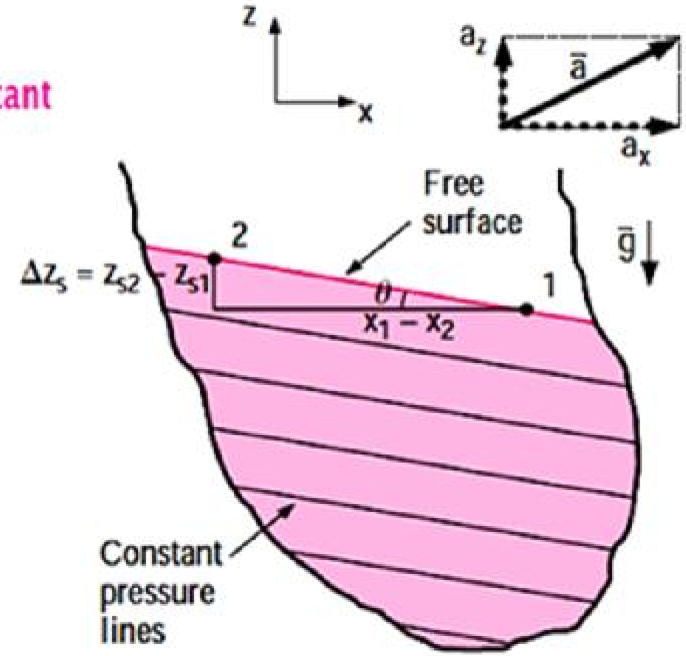
Vertical rise of surface: $$\Delta z_s = z_{s2} - z_{s1} = -\frac{a_x}{g + a_z}(x_2 - x_1)$$

- where z_s is the z-coordinate of the liquid's free surface
- The equation for surfaces of constant pressure, called **isobars,** is obtained from $dP = -\rho a_x\, dx - \rho(g + a_z)\, dz$ by setting $d_P = 0$ and replacing z by z_{isobar}, which is the z-coordinate (the vertical distance) of the surface as a function of x. It gives

Acceleration on a Straight Path

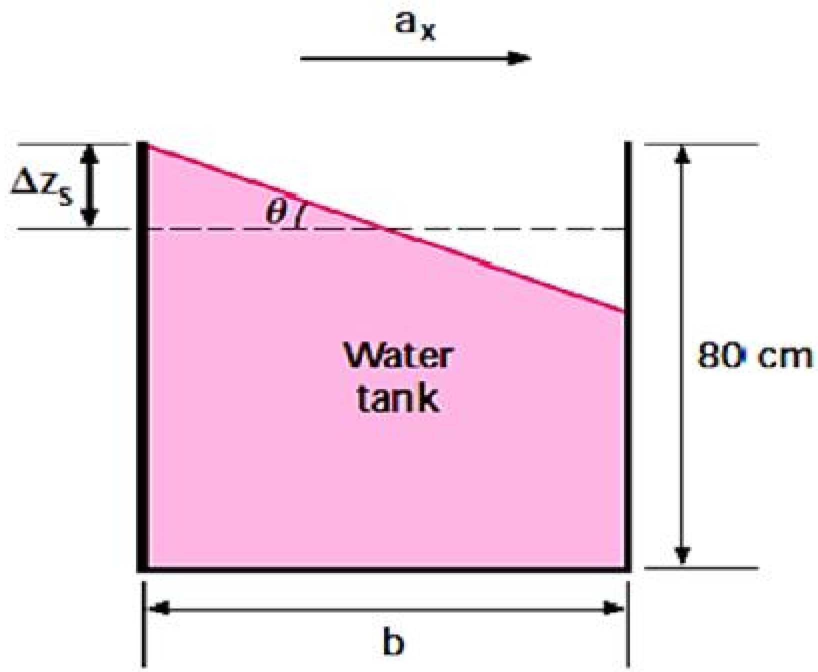
Surfaces of constant pressure: $\frac{dz_{\text{isobar}}}{dx} = -\frac{a_x}{g + a_z} = \text{constant}$

Slope of isobars: $\text{Slope} = \frac{dz_{\text{isobar}}}{dx} = -\frac{a_x}{g + a_z} = -\tan\theta$



Example. Overflow from a Water Tank During Acceleration

- An 80-cm-high fish tank of cross section 2 m x 0.6 m that is initially filled with water is to be transported on the back of a truck . The truck accelerates from 0 to 90 km/h in 10 s. If it is desired that no water spills during acceleration, determine the allowable initial water height in the tank. Would you recommend the tank to be aligned with the long or short side parallel to the direction of motion?



Example. Overflow from a Water Tank During Acceleration

SOLUTION A fish tank is to be transported on a truck. The allowable water height to avoid spill of water during acceleration and the proper orientation are to be determined.

Assumptions **1** The road is horizontal during acceleration so that acceleration has no vertical component ($a_z = 0$). **2** Effects of splashing, braking, driving over bumps, and climbing hills are assumed to be secondary and are not considered. **3** The acceleration remains constant.

Analysis We take the *x*-axis to be the direction of motion, the *z*-axis to be the upward vertical direction, and the origin to be the lower left corner of the tank. Noting that the truck goes from 0 to 90 km/h in 10 s, the acceleration of the truck is

$$a_x = \frac{\Delta V}{\Delta t} = \frac{(90 - 0)\ \text{km/h}}{10\ \text{s}}\left(\frac{1\ \text{m/s}}{3.6\ \text{km/h}}\right) = 2.5\ \text{m/s}^2$$

The tangent of the angle the free surface makes with the horizontal is

$$\tan\theta = \frac{a_x}{g + a_z} = \frac{2.5}{9.81 + 0} = 0.255 \qquad \text{(and thus } \theta = 14.3°\text{)}$$

The maximum vertical rise of the free surface occurs at the back of the tank, and the vertical midplane experiences no rise or drop during acceleration since it is a plane of symmetry. Then the vertical rise at the back of the tank relative to the midplane for the two possible orientations becomes

Case 1: The long side is parallel to the direction of motion:

$$\Delta z_{s1} = (b_1/2)\tan\theta = [(2\text{ m})/2] \times 0.255 = 0.255\text{ m} = \mathbf{25.5\ cm}$$

Case 2: The short side is parallel to the direction of motion:

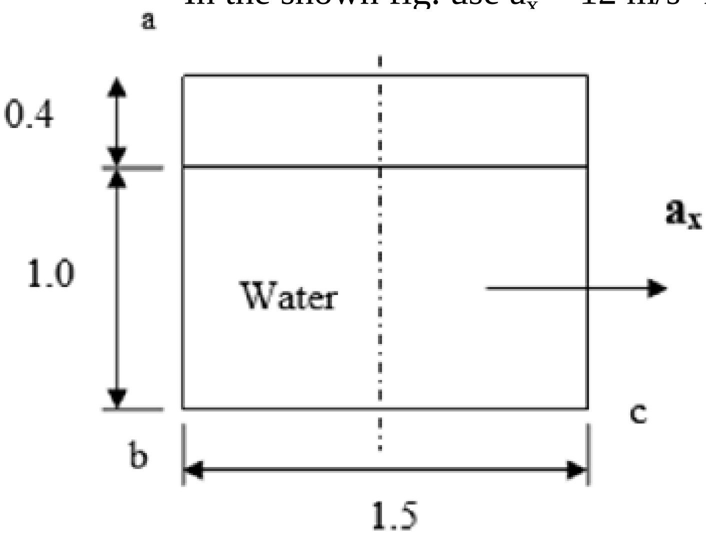
$$\Delta z_{s2} = (b_2/2)\tan\theta = [(0.6\text{ m})/2] \times 0.255 = 0.076\text{ m} = \mathbf{7.6\ cm}$$

Therefore, assuming tipping is not a problem, the tank should definitely be oriented such that its short side is parallel to the direction of motion. Emptying the tank such that its free surface level drops just 7.6 cm in this case will be adequate to avoid spilling during acceleration.

Discussion Note that the orientation of the tank is important in controlling the vertical rise. Also, the analysis is valid for any fluid with constant density, not just water, since we used no information that pertains to water in the solution.

Example:

- In the shown fig. use a_x = 12 m/s^2 find the static pressure at points a, b, and c. W=1.8 m



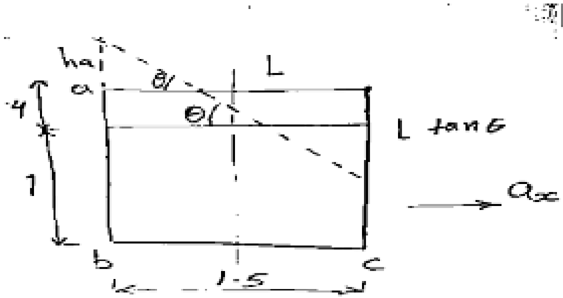
* solution:-

upon acceleration, the free surface is titled at an angle θ to the horizontal.

since $\tan\theta = -\frac{a_x}{g + a_z}$

$\therefore \tan\theta = \frac{-12}{9.81} = -1.22$

θ is in the second and fourth quarters.



volume of air before acceleration = volume of air after acceleration.

$\therefore 0.4 * 1.5 + 1 = 0.5 L (L \tan\theta)$

$0.4 * 1.5 * 1 = 0.5 L^2 * 1.22$

$L = 1$ m

$h_c = 1.4 - 1.22 = 0.18$ m

$\therefore P_c = 9810 * 0.18 = 1848.6$ Pa

$\because \tan\theta = \frac{h_a}{1.5 - L} \Rightarrow h_a = (1.5 - L)\tan\theta$

$\therefore h_a = 0.5 * 1.22 = 0.61$ m

$h_b = 1.4 + 0.61$ m

Rotation in a Cylindrical Container

We know from experience that when a glass filled with water is rotated about its axis, the fluid is forced outward as a result of the so-called centrifugal force, and the free surface of the liquid becomes concave. This is known as the *forced vortex motion*.

Container is rotating about the z-axis

$$a_r = -r\omega^2, a_\theta = a_z = 0$$

$$\frac{\partial P}{\partial r} = \rho r\omega^2, \frac{\partial P}{\partial \theta} = 0, \frac{\partial P}{\partial z} = -\rho g$$

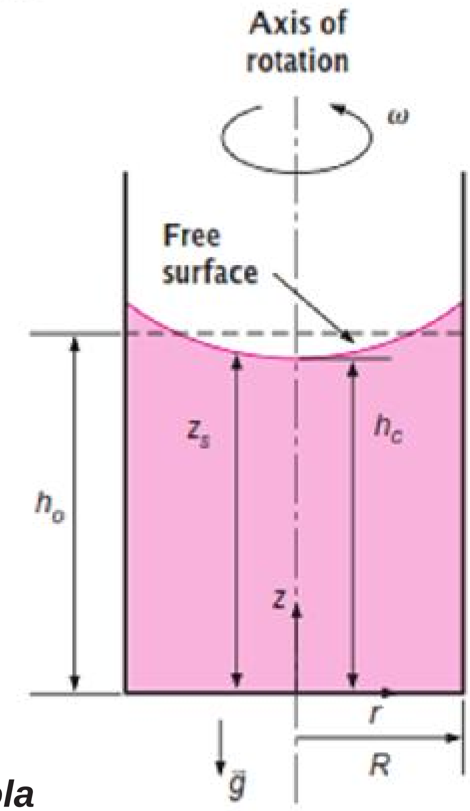
Total differential of P

$$dP = \rho r\omega^2 dr - \rho g dz$$

On an isobar, dP = 0 *Surfaces of constant pressure*

$$\frac{dz_{isobar}}{dr} = \frac{r\omega^2}{g} \rightarrow z_{isobar} = \frac{\omega^2}{2g} r^2 + C_1$$

which is the equation of a *parabola*



Rotation in a Cylindrical Container

$$z_{isobar} = \frac{\omega^2}{2g} r^2 + C_1$$

For the free surface, setting r=0 in the previous equation gives $Z_{isobar}(0)=C_1=hc$

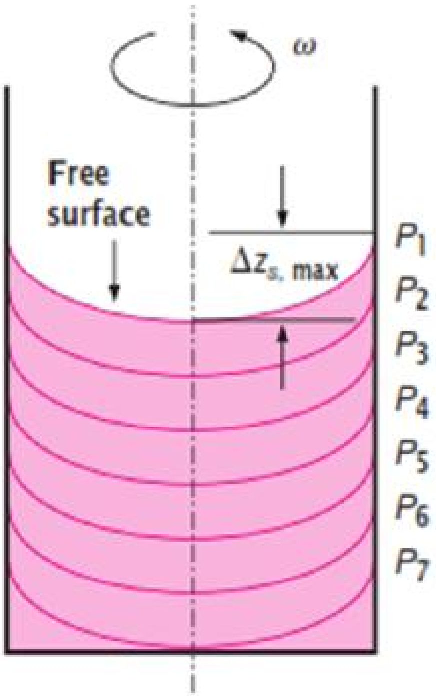
Where **hc** is the distance of the free surface from the bottom of the container along axis of rotation

Then the equation for the free surface becomes $z_s = \frac{\omega^2}{2g} r^2 + h_c$

Zs is the distance of the free surface from the bottom of the container.

The volume of a cylindrical shell element of radius *r, height Zs and thickness dr is*

$$V = \int_{r=0}^{R} 2\pi z_s r\, dr = 2\pi \int_{r=0}^{R} \left(\frac{\omega^2}{2g} r^2 + h_c\right) r\, dr = \pi R^2 \left(\frac{\omega^2 R^2}{4g} + h_c\right)$$



Since mass is conserved and density is constant, this volume must be equal to the original volume of the fluid in the container, which is

$$V = \pi R^2 h_0$$

where h_0 is the original height of the fluid in the container with no rotation. Setting these two volumes equal to each other, the height of the fluid along the centerline of the cylindrical container becomes

$$h_c = h_0 - \frac{\omega^2 R^2}{4g}$$

Then the equation of the free surface becomes

Free surface:

$$z_s = h_0 - \frac{\omega^2}{4g}(R^2 - 2r^2)$$

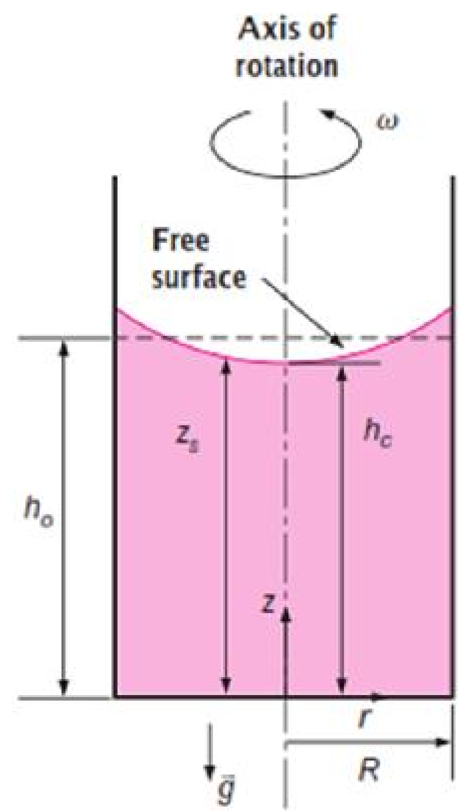
The maximum vertical height occurs at the edge where $r = R$, and the *maximum height difference* between the edge and the center of the free surface is determined by evaluating z_s at $r = R$ and also at $r = 0$, and taking their difference,

Maximum height difference:

$$\Delta z_{s,\max} = z_s(R) - z_s(0) = \frac{\omega^2}{2g}R^2$$

When ρ = constant, the pressure difference between two points 1 and 2 in the fluid is determined by integrating $dP = \rho r\omega^2\, dr - \rho g\, dz$. This yields

$$P_2 - P_1 = \frac{\rho\omega^2}{2}(r_2^2 - r_1^2) - \rho g(z_2 - z_1)$$

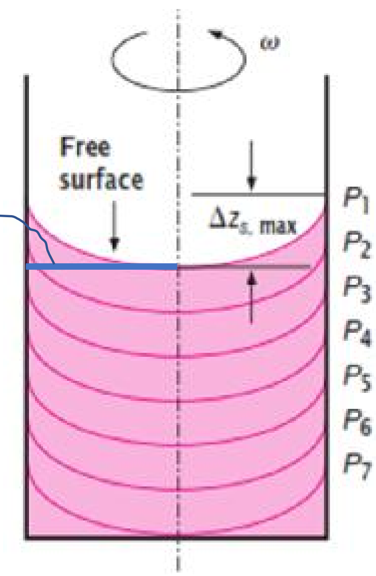


Taking point 1 to be the origin ($r = 0$, $z = 0$) where the pressure is P_0 and point 2 to be any point in the fluid (no subscript), the pressure distribution can be expressed as

Pressure variation:

$$P = P_0 + \frac{\rho\omega^2}{2} r^2 - \rho g z$$

Note that at a fixed radius, the pressure varies hydrostatically in the vertical direction, as in a fluid at rest. For a fixed vertical distance z, the pressure varies with the square of the radial distance r, increasing from the centerline toward the outer edge. In any horizontal plane, the pressure difference between the center and edge of the container of radius R is $\Delta P = \rho\omega^2 R^2/2$.



Rising of liquid during rotation

- Example:

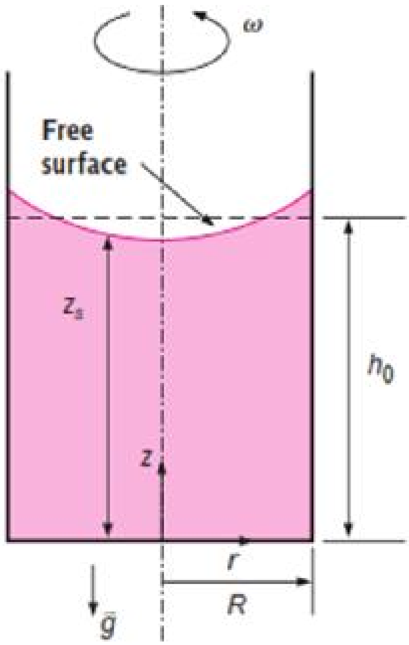
A 20-cm-diameter, 60-cm-high vertical cylindrical container, shown in Fig. 3–55, is partially filled with 50-cm-high liquid whose density is 850 kg/m³. Now the cylinder is rotated at a constant speed. Determine the rotational speed at which the liquid will start spilling from the edges of the container.

SOLUTION A vertical cylindrical container partially filled with a liquid is rotated. The angular speed at which the liquid will start spilling is to be determined.

Assumptions **1** The increase in the rotational speed is very slow so that the liquid in the container always acts as a rigid body. **2** The bottom surface of the container remains covered with liquid during rotation (no dry spots).

Analysis Taking the center of the bottom surface of the rotating vertical cylinder as the origin ($r = 0$, $z = 0$), the equation for the free surface of the liquid is given as

$$z_s = h_0 - \frac{\omega^2}{4g}(R^2 - 2r^2)$$



Rising of liquid during rotation

- Example:

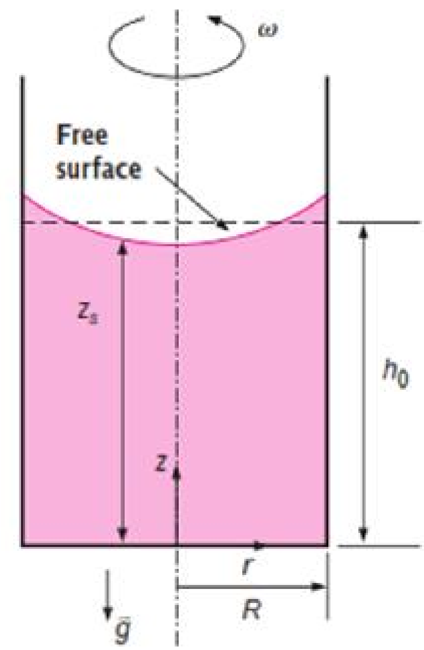
Then the vertical height of the liquid at the edge of the container where $r = R$ becomes

$$z_s(R) = h_0 + \frac{\omega^2 R^2}{4g}$$

where $h_0 = 0.5$ m is the original height of the liquid before rotation. Just before the liquid starts spilling, the height of the liquid at the edge of the container equals the height of the container, and thus $z_s(R) = 0.6$ m. Solving the

last equation for ω and substituting, the maximum rotational speed of the container is determined to be

$$\omega = \sqrt{\frac{4g[z_s(R) - h_0]}{R^2}} = \sqrt{\frac{4(9.81\ \text{m/s}^2)[(0.6 - 0.5)\ \text{m}]}{(0.1\ \text{m})^2}} = \mathbf{19.8\ rad/s}$$



Rising of liquid during rotation

- Example:

last equation for ω and substituting, the maximum rotational speed of the container is determined to be

$$\omega = \sqrt{\frac{4g[z_s(R) - h_0]}{R^2}} = \sqrt{\frac{4(9.81\ \text{m/s}^2)[(0.6 - 0.5)\ \text{m}]}{(0.1\ \text{m})^2}} = \mathbf{19.8\ rad/s}$$

Noting that one complete revolution corresponds to 2π rad, the rotational speed of the container can also be expressed in terms of revolutions per minute (rpm) as

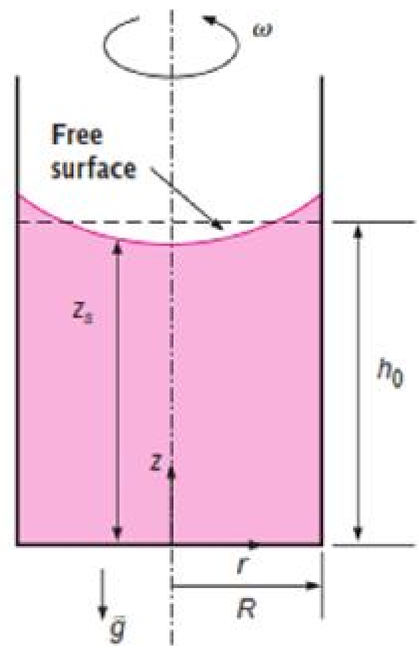
$$\dot{n} = \frac{\omega}{2\pi} = \frac{19.8\ \text{rad/s}}{2\pi\ \text{rad/rev}}\left(\frac{60\ \text{s}}{1\ \text{min}}\right) = \mathbf{189\ rpm}$$

Therefore, the rotational speed of this container should be limited to 189 rpm to avoid any spill of liquid as a result of the centrifugal effect.

Discussion Note that the analysis is valid for any liquid since the result is independent of density or any other fluid property. We should also verify that our assumption of no dry spots is valid. The liquid height at the center is

$$z_s(0) = h_0 - \frac{\omega^2 R^2}{4g} = 0.4\ \text{m}$$

Since $z_s(0)$ is positive, our assumption is validated.



Example

- An open cylindrical tank of diameter 1 m and height 1.5 m is 2/3 full of water. Find the maximum speed of rotation for which no water spills. Find the speed of rotation for which the center of the bottom starts to clear.

Upon rotation the free-surface takes the shape of a paraboloid
For maximum speed with no spill:
The height of the paraboloid surface is 1 m

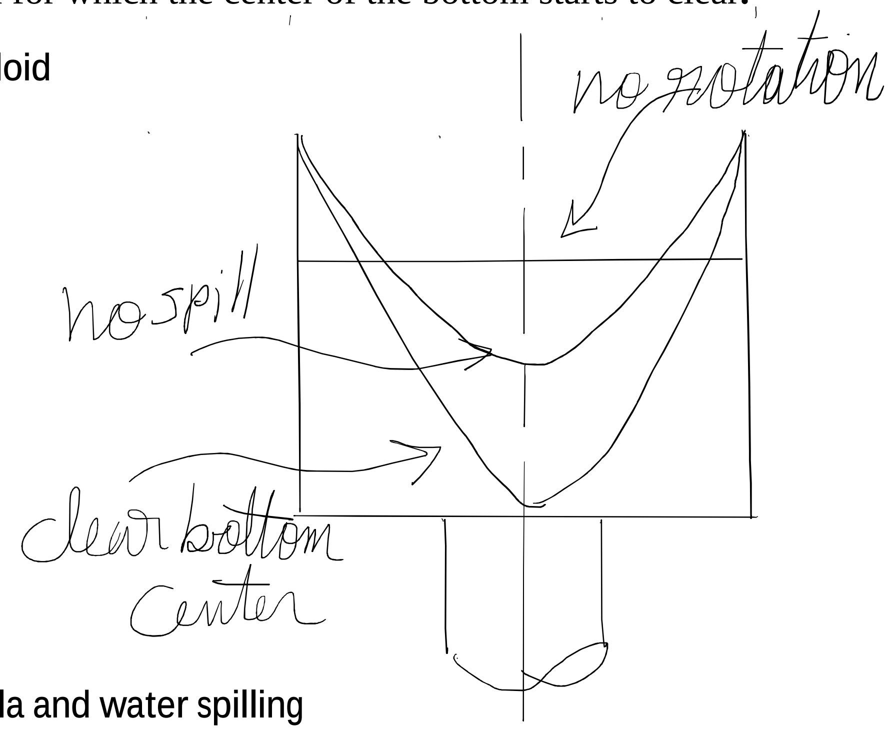
$$Z_2 - Z_1 = \frac{\omega^2}{2g}(r_2^2 - r_1^2)$$

The previous equation comes from

$$Z_s = h_0 - \frac{\omega^2}{4g}(R^2 - 2r^2)$$

$$1 = \frac{\omega^2}{2 * 9.81}(0.5^2 - 0) \Rightarrow \omega = 8.859\, rad/sec$$

$$\text{N} = 84.6\, r.p.m$$



Any further increase of rpm will increase the depth of parabola and water spilling

The center of the tank bottom starts to clear when the depth of the parabola is 1.5 m and w=10..85 rad/s

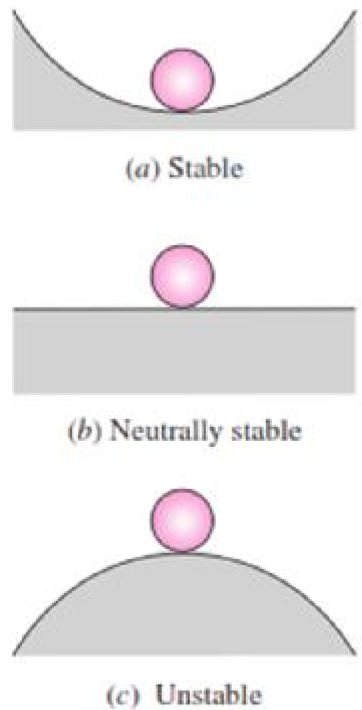
Stability of immersed and floating bodies

- Stability of instability concepts

Case (*a*) *is stable* ***since any small disturbance (someone moves the ball*** to the right or left) generates a restoring force (due to gravity) that returns it to its initial position.

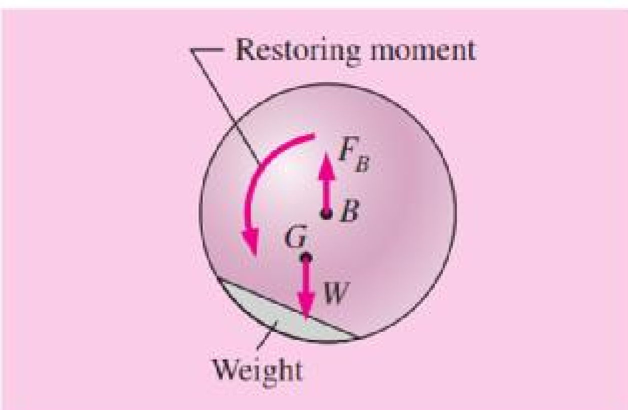
Case (*b*) *is neutrally stable* ***because if someone moves*** the ball to the right or left, it will stay put at its new location. It has no tendency to move back to its original location, nor does it continue to move away.

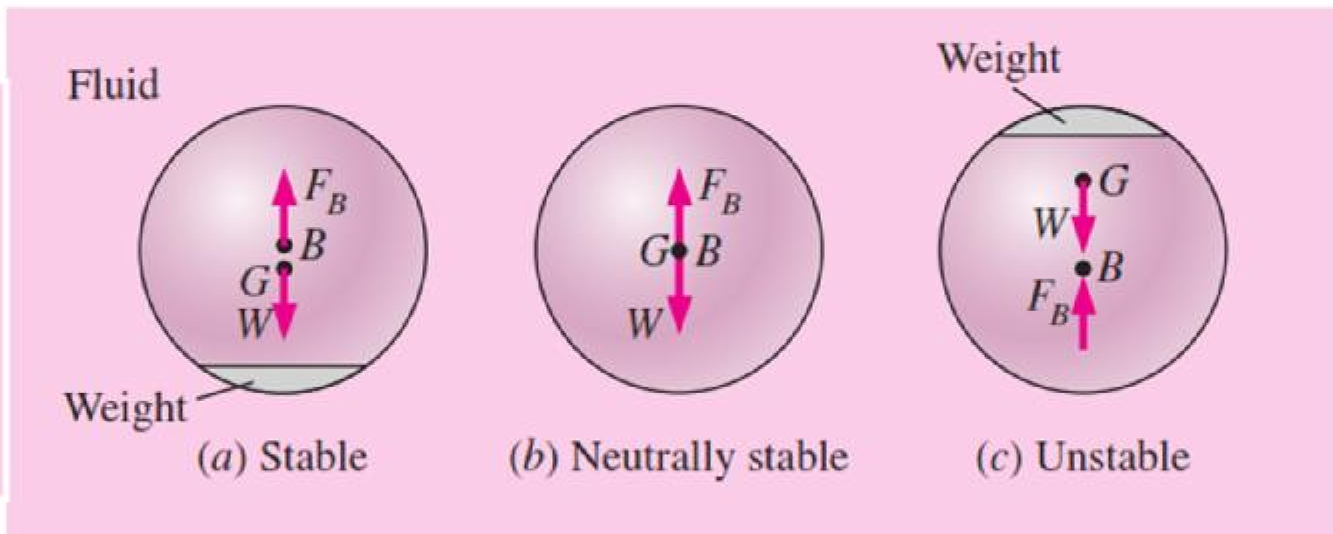
Case (*c*) *is a situation in which the ball may be at rest at the moment,* but any disturbance, even an infinitesimal one, causes the ball to roll off the hill—it does not return to its original position; rather it *diverges from it. This* situation is unstable.



Stability of immersed and floating bodies

- The *rotational stability of an immersed body* depends on the relative locations of the *center of gravity G* of the body and the *center of buoyancy B*, which is the centroid of the displaced volume.

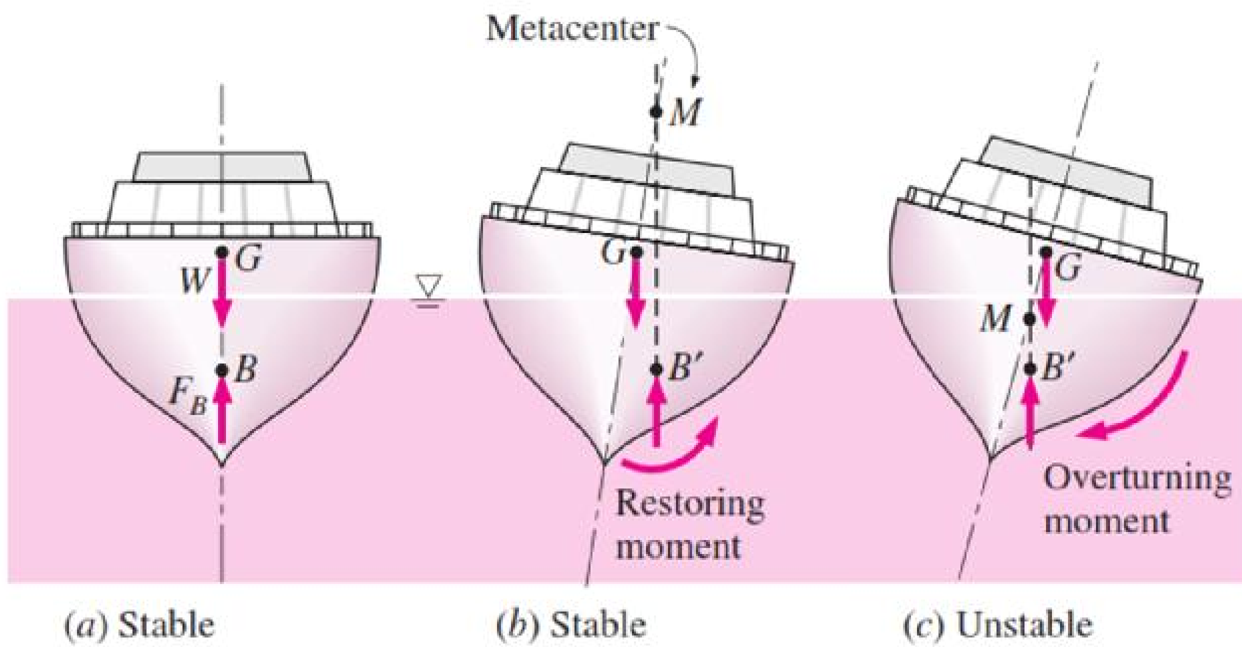




Stability of immersed and floating bodies

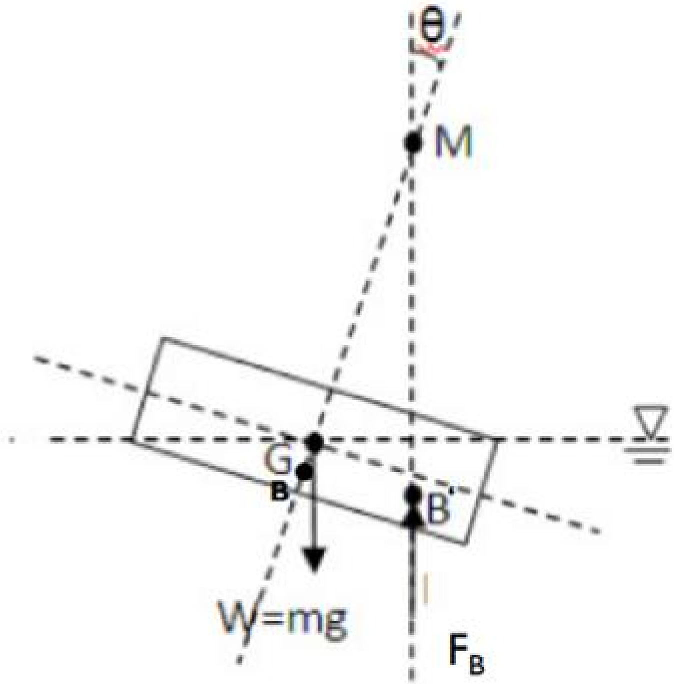
- Stability of floating bodies
  - The measure of stability for floating bodies is the *metacentric height GM.*

Point M: the intersection point of the lines of action of the buoyant force through the body before and after rotation.



Stability of immersed and floating bodies

- **Metacentric height (GM):** The distance between the center of gravity (G) of floating body and the metacenter (M) is called metacentric height. (i.e., distance GM shown in fig)



GM=BM-BG

Stability of immersed and floating bodies

- Example :
- A solid cylinder 2 m in diameter and 2 m high is floating in water with its axis vertical. If the specific gravity of the material of cylinder is 0.65 find its metacentric height. State also whether the equilibrium is stable or unstable.

Size of solid cylinder= 2m dia, & 2m height

Specific gravity solid cylinder=0.65

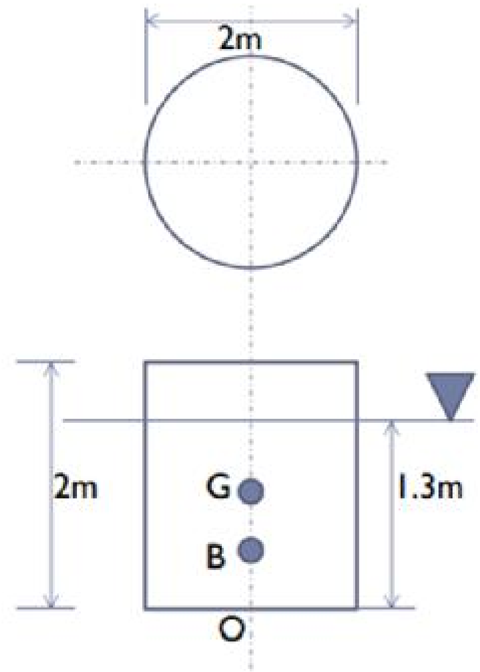
Let h is depth of immersion=?

For equilibrium

Weight of water displaced = weight of wooden block

$9.81(\pi/4(2)^2(h))=9.81(0.65).(\pi/4(2)^2(2))$

$h=0.65(2)=1.3m$



Stability of immersed and floating bodies

- Example :

Center of buoyancy from O=OB=1.3/2=0.65m

Center of gravity from O=OG=2/2=1m

BG=1-0.65=0.35m

Also; BM=I/V

Moment of inertia=I=$(\pi/64)(2)^4=0.785m^4$

Volume displaced=V=$(\pi/4)(2)^4(1.3)=4.084m^3$

BM=I/V=0.192m

GM=BM-BG=0.192-0.35=-0.158m

-ve sign indicate that the metacenter (M) is below the center of gravity (G), therefore, the cylinder is in **unstable equilibrium**